

SEMI-INFINITE SCHUBERT VARIETIES AND QUANTUM K -THEORY OF FLAG MANIFOLDS

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ABSTRACT. Let \mathfrak{g} be a semi-simple Lie algebra over \mathbb{C} and let $\mathcal{B}_{\mathfrak{g}}$ be its flag variety. In this paper we study the spaces $Z_{\mathfrak{g}}^{\alpha}$ of based quasi-maps $\mathbb{P}^1 \rightarrow \mathcal{B}_{\mathfrak{g}}$ (introduced in [17]) as well as their affine versions (corresponding to \mathfrak{g} being untwisted affine algebra) introduced in [5]. The purpose of this paper is two-fold. First we study the singularities of the above spaces (as was explained in [17] and [3] they are supposed to model singularities of the not rigorously defined “semi-infinite Schubert varieties”). We show that $Z_{\mathfrak{g}}^{\alpha}$ is normal and when \mathfrak{g} is simply laced, $Z_{\mathfrak{g}}^{\alpha}$ is Gorenstein and has rational singularities; some weaker results are proved also in the affine case.

The second purpose is to study the character of the ring of functions on $Z_{\mathfrak{g}}^{\alpha}$. When \mathfrak{g} is finite-dimensional and simply laced we show that the generating function of these characters satisfies the “fermionic formula” version of quantum difference Toda equation (cf. [14]), thus extending the results for $\mathfrak{g} = \mathfrak{sl}(N)$ from [19] and [6]; in view of the first part this also proves a conjecture from [19] describing the quantum K -theory of $\mathcal{B}_{\mathfrak{g}}$ in terms of the Langlands dual quantum group $U_q(\check{\mathfrak{g}})$ (for non-simply laced \mathfrak{g} certain modification of that conjecture is necessary). Similar analysis (modulo certain assumptions) is performed for affine \mathfrak{g} , extending the results of [6].

1. INTRODUCTION

1.1. Spaces of quasi-maps. Let G be a semi-simple simply connected group over \mathbb{C} with Lie algebra \mathfrak{g} ; we shall denote by $\check{\mathfrak{g}}$ the Langlands dual algebra of \mathfrak{g} . Let also $\mathcal{B}_{\mathfrak{g}}$ denote its flag variety. We have $H_2(\mathcal{B}_{\mathfrak{g}}, \mathbb{Z}) = \Lambda$, the coroot lattice of \mathfrak{g} . We shall denote by Λ_+ the sub-semigroup of positive elements in Λ .

Let $\mathbf{C} \simeq \mathbb{P}^1$ denote a (fixed) smooth connected projective curve (over \mathbb{C}) of genus 0; we are going to fix a marked point $\infty \in \mathbf{C}$. For each $\alpha \in \Lambda_+$ we can consider the space $\mathcal{M}_{\mathfrak{g}}^{\alpha}$ of maps $\mathbf{C} \rightarrow \mathcal{B}_{\mathfrak{g}}$ of degree α . This is a smooth quasi-projective variety. It has a compactification $\mathcal{QM}_{\mathfrak{g}}^{\alpha}$ by means of the space of *quasi-maps* from \mathbf{C} to $\mathcal{B}_{\mathfrak{g}}$ of degree α . Set-theoretically this compactification looks can be described as follows:

$$\mathcal{QM}_{\mathfrak{g}}^{\alpha} = \bigsqcup_{0 \leq \beta \leq \alpha} \mathcal{M}_{\mathfrak{g}}^{\beta} \times \text{Sym}^{\alpha-\beta}(\mathbf{C}) \quad (1.1)$$

where $\text{Sym}^{\alpha-\beta}(\mathbf{C})$ stands for the space of “colored divisors” of the form $\sum \gamma_i x_i$ where $x_i \in \mathbf{C}$, $\gamma_i \in \Lambda_+$ and $\sum \gamma_i = \alpha - \beta$.

Let us fix a pair of opposite Borel subgroups $B, B_- \subset G$; then we can write $\mathcal{B}_{\mathfrak{g}} = G/B$. We can now consider the space $\overset{\circ}{Z}_{\mathfrak{g}}^{\alpha}$ of *based maps* $(\mathbf{C}, \infty) \rightarrow (\mathcal{B}_{\mathfrak{g}} = G/B, e_-)$ (here $e_- \in G/B$ denotes the class of B_- , and a map $f : \mathbf{C} \rightarrow \mathcal{B}_{\mathfrak{g}}$ is called based if $f(\infty) = e_-$). This is a quasi-affine variety; the corresponding space $Z_{\mathfrak{g}}^{\alpha}$ of based quasi-maps (a.k.a. Zastava space

in the terminology of [17] and [13]) is affine. It possesses a stratification similar to (1.1) but with \mathbf{C} in the right hand side of (1.1) replaced by $\mathbf{C} - \infty$.

The following theorem is the first main result of this paper:

Theorem 1.2. (1) For any \mathfrak{g} and α the schemes $Z_{\mathfrak{g}}^{\alpha}$ and $\mathcal{QM}_{\mathfrak{g}}^{\alpha}$ are normal.
(2) Assume that \mathfrak{g} is simply laced. Then $Z_{\mathfrak{g}}^{\alpha}$ (and $\mathcal{QM}_{\mathfrak{g}}^{\alpha}$) is Gorenstein (in particular, Cohen-Macaulay) and has canonical (hence rational) singularities.

1.3. Connection to quantum K -theory of $\mathcal{B}_{\mathfrak{g}}$. In fact, we believe that $Z_{\mathfrak{g}}^{\alpha}$ must have rational singularities for all \mathfrak{g} (not necessarily simply laced). Let us explain the importance of this assertion. Recall, that a scheme Z has rational singularities, if for some (equivalently, for any) resolution $\pi : \tilde{Z} \rightarrow Z$ we have $R\pi_*(\mathcal{O}_{\tilde{Z}}) = \mathcal{O}_Z$. The scheme $Z_{\mathfrak{g}}^{\alpha}$ has a resolution by means of the Kontsevich moduli space $M_{\mathfrak{g}}^{\alpha}$ of stable maps from a nodal curve C of genus 0 to $\mathcal{B}_{\mathfrak{g}} \times \mathbb{P}^1$ which have degree $(\alpha, 1)$ and with some analog of the “based” condition (cf. Section 5 for more detail). The space $M_{\mathfrak{g}}^{\alpha}$ is a smooth Deligne-Mumford stack which has a natural action of $T \times \mathbb{C}^*$, where $T \subset B \subset G$ is a maximal torus (here the action of T comes from the fact that it acts on $\mathcal{B}_{\mathfrak{g}}$ preserving e and the action of \mathbb{C}^* comes from the action on \mathbb{P}^1 preserving ∞). It is shown in [19] that the $T \times \mathbb{C}^*$ -equivariant pushforward of $\mathcal{O}_{M_{\mathfrak{g}}^{\alpha}}$ to $\text{Spec}(\mathbb{C})$ (i.e. the character of $[R\Gamma(M_{\mathfrak{g}}^{\alpha}, \mathcal{O}_{M_{\mathfrak{g}}^{\alpha}})]$ of $R\Gamma(M_{\mathfrak{g}}^{\alpha}, \mathcal{O}_{M_{\mathfrak{g}}^{\alpha}})$ with respect to $T \times \mathbb{C}^*$) is exactly the object that one needs to compute when studying quantum K -theory of $\mathcal{B}_{\mathfrak{g}}$. Theorem 1.2 implies that (for simply laced \mathfrak{g}) one can replace this equivariant pushforward with the character $[\mathcal{O}_{Z_{\mathfrak{g}}^{\alpha}}]$ of the ring of polynomial functions on $Z_{\mathfrak{g}}^{\alpha}$ with respect to the action of $T \times \mathbb{C}^*$. In what follows we shall denote this character by \mathfrak{J}_{α} ; this is a rational function on $T \times \mathbb{C}^*$ and we are going to write $\mathfrak{J}_{\alpha} = \mathfrak{J}_{\alpha}(z, q)$ where $z \in T, q \in \mathbb{C}^*$.

1.4. Fermionic formula. The characters $[R\Gamma(M_{\mathfrak{g}}^{\alpha}, \mathcal{O}_{M_{\mathfrak{g}}^{\alpha}})]$ were studied in [19] for $G = \text{SL}(N)$ and their generating function was shown to be an eigen-function of the *quantum difference Toda integrable system* (cf. [11], [26]); this result was reproved in [6] using other methods. It was conjectured in *loc. cit.* that the same result should hold for any \mathfrak{g} . In this paper we are going to prove the following:

Theorem 1.5. Assume that \mathfrak{g} is simply laced. Then the functions \mathfrak{J}_{α} satisfy the following recursive relation:

$$\mathfrak{J}_{\alpha} = \sum_{0 \leq \beta \leq \alpha} \frac{q^{(\beta, \beta)/2} z^{\beta^*}}{(q)_{\alpha - \beta}} \mathfrak{J}_{\beta}. \quad (1.2)$$

Here $\beta \mapsto \beta^*$ stands for the natural isomorphism between the coroot lattice of \mathfrak{g} and its root lattice.

The equation (1.2) is not new: it appears in [14], where the authors show that (1.2) holds precisely if and only if the function

$$J_{\mathfrak{g}}(z, x, q) = \sum_{\alpha \in \Lambda_+} x^{\alpha} \mathfrak{J}_{\alpha} \quad (1.3)$$

is an eigen-function of the above-mentioned quantum difference Toda system (here x lies in the dual torus \check{T}); the generating function (1.3) is called *the equivariant K -theoretic J -function* of $\mathcal{B}_{\mathfrak{g}}$ in [19] (of course, in [19] the authors use $[R\Gamma(M_{\mathfrak{g}}^{\alpha}, \mathcal{O}_{M_{\mathfrak{g}}^{\alpha}})]$ instead of \mathfrak{J}_{α}

but due to Theorem 1.2 we know that they are the same). Thus Theorem 1.5 and the main result of [14] imply the following:

Corollary 1.6. *Let \mathfrak{g} be simply laced. Then the equivariant K -theoretic J -function of $\mathcal{B}_{\mathfrak{g}}$ is an eigen-function of the quantum difference Toda integrable system associated with \mathfrak{g} .*

One may ask whether the assumption that \mathfrak{g} is simply laced is really essential. It is easy to see that verbatim Theorem 1.5 (and thus also Corollary 1.6) does not hold for non-simply laced \mathfrak{g} . On the other hand in Section 8 we show how to modify the problem a little (using the recent result of [27]) in order to make a correct statement for all \mathfrak{g} . It is worthwhile to note that the corresponding analog of $J_{\mathfrak{g}}$ in that case becomes an eigen-function of the quantum difference Toda system associated with $\check{\mathfrak{g}}$ (in the simply laced case we have $\mathfrak{g} = \check{\mathfrak{g}}$). The reader should compare this statement with the main result of [21] which deals with the “usual” (i.e. cohomological) J -function of $\mathcal{B}_{\mathfrak{g}}$.

1.7. Representation-theoretic interpretation. In this subsection we discuss possible interpretation of the above results in terms of geometric representation theory; this subsection will not be used in the future, so uninterested reader may skip this discussion and go to Section 1.9.

Corollary 1.6 and the constructions of [26] and [14] imply also the following:

Corollary 1.8. *In the simply laced case the function $J_{\mathfrak{g}}$ is equal to the Whittaker matrix coefficient in the universal Verma of $U_q(\check{\mathfrak{g}})$.*

In [6] this result was proved directly for $G = \mathrm{SL}(N)$. Namely, in that case the space $Z_{\mathfrak{g}}^{\alpha}$ has a small resolution of singularities (usually called Laumon’s resolution) which we shall denote by \mathcal{P}^{α} . In [6] we construct an action of the quantum group $U_q(\mathfrak{sl}(N))$ on $\mathcal{V} = \bigoplus_{\alpha} K_{T \times \mathbb{C}^*}(\mathcal{P}^{\alpha})_{\mathrm{loc}}$ (here the subscript “loc” means “localized equivariant K -theory”) and identify the corresponding $U_q(\mathfrak{sl}(N))$ module with the *universal Verma module*. Moreover, the natural pairing on \mathcal{V} gets identified with the Shapovalov form on the Verma module. In addition if we denote by $1_{\alpha} \in K_{T \times \mathbb{C}^*}(\mathcal{P}^{\alpha})$ then the formal sum $\sum_{\alpha} 1_{\alpha}$ (lying in some completion of \mathcal{V}) is the Whittaker vector in \mathcal{V} (i.e. an eigen-vector of the positive part of $U_q(\mathfrak{sl}(N))$). It is easy to see that these results imply Corollary 1.8 (we refer the reader to [6] for the details).

It would be very interesting to prove Corollary 1.8 along similar lines, however we don’t know how to do this, since for general \mathfrak{g} there is no resolution of $Z_{\mathfrak{g}}^{\alpha}$ similar to \mathcal{P}^{α} . In addition we would like to mention that the notion of Whittaker vector for \mathfrak{g} (or $\check{\mathfrak{g}}$), which is developed in [26] (cf. [11] for a closely related approach) depends on a choice of orientation of the Dynkin diagram of \mathfrak{g} ; it would be very interesting to understand how it can be incorporated in the above constructions (for $\mathfrak{g} = \mathfrak{sl}(N)$ there is a natural choice of orientation).

1.9. Idea of the proof of normality. Let us now go back and explain the idea of the proof of Theorem 1.2(1), since in our opinion this proof is of independent interest.

Let $\mathrm{Gr}_G = G((t))/G[[t]]$ be the affine Grassmannian of G . It is well-known that the orbits of $G[[t]]$ on Gr_G are in one-to-one correspondence with the elements of the dominant cone Λ^+ ; for each $\lambda \in \Lambda^+$ we shall denote the corresponding orbit by Gr_G^{λ} . Its closure $\overline{\mathrm{Gr}}_G^{\lambda}$ is the union of all Gr_G^{μ} with $\mu \leq \lambda$. It is well-known (cf. e.g. [12]) that $\overline{\mathrm{Gr}}_G^{\lambda}$ is normal, Cohen-Macaulay and has rational singularities (in fact, it is also Gorenstein - cf. [8]).

The schemes $Z_{\mathfrak{g}}^{\alpha}$ were originally defined in [17] in order to give a model for the singularities of $\overline{\text{Gr}}_G^{\lambda}$ at a point of Gr_G^{μ} when both λ and μ are very large and $\lambda - \mu = \alpha$. However, although this statement was used as a guiding principle in many works related to $Z_{\mathfrak{g}}^{\alpha}$ (cf. [3] for a review), it was never given any precise meaning.

The purpose of Section 2 is to formulate some version of the above principle precisely. This formulation immediately implies normality of $Z_{\mathfrak{g}}^{\alpha}$ but other parts of Theorem 1.2 still have to be proven by other means. Roughly speaking we show the following. Given λ and μ as above one can construct certain transversal slice $\overline{W}_{G,\mu}^{\lambda}$ to Gr_G^{μ} in $\overline{\text{Gr}}_G^{\lambda}$. This transversal slice is also acted on by $T \times \mathbb{C}^*$. In Section 2 we construct a $T \times \mathbb{C}^*$ -equivariant map $\overline{W}_{G,\lambda-\alpha}^{\lambda} \rightarrow Z_{\mathfrak{g}}^{\alpha}$ and we show that this map induces an isomorphism on functions of given homogeneity degree with respect to \mathbb{C}^* when λ is very large. This easily implies that $Z_{\mathfrak{g}}^{\alpha}$ is normal.

1.10. Affine case. The definition of the schemes $Z_{\mathfrak{g}}^{\alpha}$ was generalized in [5] to the case when \mathfrak{g} is an untwisted affine algebra. We conjecture that Theorem 1.2 and Theorem 1.5 hold in this case; this should be useful for studying the Nekrasov partition function of 5-dimensional pure gauge theory compactified on S^1 (cf. [25]) in the spirit of [4]. In Section 3 we prove Theorem 1.2 for $\mathfrak{g} = \mathfrak{sl}(N)_{\text{aff}}$; this easily implies Theorem 1.5 in this case, in view of the results of [6].

1.11. Contents. This paper is organized as follows. In Section 2 we discuss the relation between $Z_{\mathfrak{g}}^{\alpha}$ and the transversal slices $\overline{W}_{G,\mu}^{\lambda}$ in the affine Grassmannian and prove that the schemes $Z_{\mathfrak{g}}^{\alpha}$ are normal. In Section 3 we use a different method to show that the affine analogs of $Z_{\mathfrak{g}}^{\alpha}$ are normal, Gorenstein and have rational singularities for $G = \text{SL}(N)$. In Section 4 we study the equation of the boundary of $Z_{\mathfrak{g}}^{\alpha}$; we use it in Section 5 in order to prove the second part of Theorem 1.2. Section 6 is devoted to proving that $\partial_{\gamma} Z_{\mathfrak{g}}^{\alpha}$ is Cohen-Macaulay; this result is used in Section 7 in order to prove Theorem 1.5. Finally in Section 8 we explain how to extend Theorem 1.2 and Theorem 1.5 to non-simply laced case using the twisted affine Grassmannian studied in [27].

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2. NORMALITY OF ZASTAVA SPACES VIA TRANSVERSAL SLICES IN THE AFFINE GRASSMANNIAN

2.1. Quasi-maps and Zastava spaces. In this section we recall the definition of \mathcal{QM}_g^α and Z_g^α . Choose a Borel subgroup $B \subset G$ with unipotent radical U . The quotient G/U is a quasi-affine variety and we denote by $\overline{G/U}$ its affine closure. The torus $T = B/U$ acts on G/U on the right and this action extends to $\overline{G/U}$.

Let \mathcal{F}_T be a T -bundle over \mathbf{C} . For every weight $\check{\lambda} : T \rightarrow \mathbb{G}_m$ of \mathbf{T} we may consider the associated line bundle $\mathcal{F}_T^{\check{\lambda}}$ on \mathbf{C} . We say that \mathcal{F}_T has degree $\alpha \in \Lambda$ if for every $\check{\lambda}$ as above the bundle $\mathcal{F}_T^{\check{\lambda}}$ has degree $\langle \check{\lambda}, \alpha \rangle$. Then the scheme \mathcal{QM}_g^α parameterizes the following data:

- a) T -bundle \mathcal{F}_T on \mathbf{C} of degree $-\alpha$;
- b) A T -equivariant map $\kappa : \mathcal{F}_T \rightarrow \mathbf{C} \times \overline{G/U}$ such that over generic point of \mathbf{C} this map goes to $\mathbf{C} \times G/U$.

More explicitly, giving κ is equivalent to specifying the following data: for every dominant $\check{\lambda}$ the map we need to specify an embedding of locally free sheaves $\kappa^{\check{\lambda}} : \mathcal{F}_T^{\check{\lambda}} \rightarrow \mathcal{O}_{\mathbf{C}} \otimes V(\check{\lambda})$ (where $V(\check{\lambda})$ denotes the corresponding irreducible representation of G). The maps $\kappa^{\check{\lambda}}$ must satisfy certain *Plücker relations*; we refer the reader to [3] for the details. It is easy to make the above into a modular problem, which defines \mathcal{QM}_g^α as a scheme, which is reduced, irreducible and projective of dimension $2|\alpha| + \dim(G/B)$ (here $|\alpha| = \langle \check{\rho}_g, \alpha \rangle$, where $\check{\rho}_g$ denotes the half-sum of the positive roots of \mathfrak{g}).

Given (\mathcal{F}_T, κ) as above let $\mathbf{U} \subset \mathbf{C}$ be the open subset of \mathbf{C} over which κ lands in $\mathbf{C} \times G/U$. If $x \in \mathbf{U}$ we shall say that the quasi-map (\mathcal{F}_T, κ) has no defect at x . It is clear that κ defines a map $f : \mathbf{U} \rightarrow G/B$. The (open dense) subset of \mathcal{QM}_g^α consisting of those quasi-maps for which $\mathbf{U} = \mathbf{C}$ is precisely the space \mathcal{M}_g^α of maps $f : \mathbf{C} \rightarrow \mathcal{B}_g = G/B$ of degree α .

Let us now fix another Borel subgroup B_- such that $B \cap B_- \simeq T$; then B_- defines a point $e_- \in \mathcal{B}_g = G/B$. The scheme Z_g^α is a locally closed subscheme of \mathcal{QM}_g^α which corresponds to those quasi-maps which have no defect at $\infty \in \mathbf{C}$ and such that $f(\infty) = e_-$ where f is as above. The scheme Z_g^α is an affine, reduced and irreducible scheme of dimension $2|\alpha|$. The intersection $\mathring{Z}_g^\alpha = Z_g^\alpha \cap \mathcal{M}_g^\alpha$ is the space of *based maps* $f : (\mathbf{C}, \infty) \rightarrow (\mathcal{B}_g, e_-)$ (i.e. those maps which send ∞ to e_-).

The schemes \mathcal{QM}_g^α and Z_g^α possess the following stratification:

$$\mathcal{QM}_g^\alpha = \bigsqcup_{0 \leq \beta \leq \alpha} \mathcal{M}_g^\beta \times \text{Sym}^{\alpha-\beta}(\mathbf{C}); \quad Z_g^\alpha = \bigsqcup_{0 \leq \beta \leq \alpha} \mathring{Z}_g^\beta \times \text{Sym}^{\alpha-\beta}(\mathbf{C} - \infty). \quad (2.1)$$

Here for any curve X and $\gamma \in \Lambda_+$ we denote by $\text{Sym}^\gamma(X)$ the scheme parameterizing “colored divisors” $\sum \gamma_i x_i$ where $x_i \in X$, $\gamma_i \in \Lambda_+$ and $\sum \gamma_i = \gamma$.

2.2. The affine Grassmannian. Let $\mathcal{K} = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$. By the *affine Grassmannian* of G we will mean the quotient $\text{Gr}_G = G(\mathcal{K})/G(\mathcal{O})$. It is known (cf. [1, 24]) that Gr_G is the set of \mathbb{C} -points of an ind-scheme over \mathbb{C} , which we will denote by the same symbol.

Since G is simply connected, its coweight lattice coincides with the coroot lattice $\Lambda = \Lambda_G$. We will denote the cone of dominant coweights by $\Lambda^+ \subset \Lambda$. Let Λ^\vee denote the dual lattice (this is the weight lattice of G). We let $2\check{\rho}_G$ denote the sum of the positive roots of G .

The group-scheme $G(\mathcal{O})$ acts on Gr_G on the left and its orbits can be described as follows. One can identify the lattice Λ_G with the quotient $T(\mathcal{K})/T(\mathcal{O})$. Fix $\lambda \in \Lambda_G$ and let s^λ denote

any lift of λ to $T(\mathcal{K})$. Let Gr_G^λ denote the $G(\mathcal{O})$ -orbit of s^λ (this is clearly independent of the choice of $\lambda(s)$). The following result is well-known:

Lemma 2.3. (1)

$$\text{Gr}_G = \bigcup_{\lambda \in \Lambda_G} \text{Gr}_G^\lambda.$$

(2) We have $\text{Gr}_G^\lambda = \text{Gr}_G^\mu$ if and only if λ and μ belong to the same W -orbit on Λ_G (here W is the Weyl group of G). In particular,

$$\text{Gr}_G = \bigsqcup_{\lambda \in \Lambda_G^+} \text{Gr}_G^\lambda.$$

(3) For every $\lambda \in \Lambda^+$ the orbit Gr_G^λ is finite-dimensional and its dimension is equal to $\langle \lambda, 2\check{\rho}_{\mathfrak{g}} \rangle$.

Let $\overline{\text{Gr}}_G^\lambda$ denote the closure of Gr_G^λ in Gr_G ; this is an irreducible projective algebraic variety; one has $\text{Gr}_G^\mu \subset \overline{\text{Gr}}_G^\lambda$ if and only if $\lambda - \mu$ is a sum of positive roots of \check{G} .

2.4. Transversal slices. Consider the group $G[t^{-1}] \subset G((t))$; let us denote by $G[t^{-1}]_1$ the kernel of the natural (“evaluation at ∞ ”) homomorphism $G[t^{-1}] \rightarrow G$. For any $\lambda \in \Lambda$ let $\text{Gr}_{G,\lambda} = G[t^{-1}] \cdot t^\lambda$. Then it is easy to see that one has

$$\text{Gr}_G = \bigsqcup_{\lambda \in \Lambda^+} \text{Gr}_{G,\lambda}$$

Let also $\mathcal{W}_{G,\lambda}$ (resp. $\mathcal{W}_{G,\lambda}$) denote the $G[t^{-1}]_1$ -orbit (resp. $G[t^{-1}]$ -orbit) of t^λ . For any $\lambda, \mu \in \Lambda^+$, $\lambda \geq \mu$ set

$$\text{Gr}_{G,\mu}^\lambda = \text{Gr}_G^\lambda \cap \text{Gr}_{G,\mu}, \quad \overline{\text{Gr}}_{G,\mu}^\lambda = \overline{\text{Gr}}_G^\lambda \cap \text{Gr}_{G,\mu}$$

and

$$\mathcal{W}_{G,\mu}^\lambda = \text{Gr}_G^\lambda \cap \mathcal{W}_{G,\mu}, \quad \overline{\mathcal{W}}_{G,\mu}^\lambda = \overline{\text{Gr}}_G^\lambda \cap \mathcal{W}_{G,\mu}.$$

Note that $\overline{\mathcal{W}}_{G,\mu}^\lambda$ contains the point t^μ in it.

Lemma 2.5. (1) The point t^μ is the only \mathbb{C}^* -fixed point in $\overline{\mathcal{W}}_{G,\mu}^\lambda$. The action of \mathbb{C}^* on $\overline{\mathcal{W}}_{G,\mu}^\lambda$ is “repelling”, i.e. for any $w \in \overline{\mathcal{W}}_{G,\mu}^\lambda$ we have $\lim_{a \rightarrow \infty} a(w) = t^\mu$.

(2) The orbit $G \cdot t^\mu$ is a connected component of the \mathbb{C}^* -fixed point set $\text{Gr}_G^{\mathbb{C}^*}$, isomorphic to a partial flag variety of G . The action of \mathbb{C}^* on $\mathcal{W}_{G,\mu}$ is “repelling”, i.e. for any $w \in \mathcal{W}_{G,\mu}$ we have $\lim_{a \rightarrow \infty} a(w) \in G \cdot t^\mu$.

(3) There exists an open subset \mathcal{U} in Gr_G^μ and an open embedding $\mathcal{U} \times \overline{\mathcal{W}}_{G,\mu}^\lambda \hookrightarrow \overline{\text{Gr}}_G^\lambda$ such that the diagram

$$\begin{array}{ccc} \mathcal{U} \times \{s^\mu\} & \longrightarrow & \text{Gr}_G^\mu \times \{t^\mu\} \\ \downarrow & & \downarrow \\ \mathcal{U} \times \overline{\mathcal{W}}_{G,\mu}^\lambda & \longrightarrow & \overline{\text{Gr}}_G^\lambda \end{array}$$

is commutative. In other words, $\overline{\mathcal{W}}_{G,\mu}^\lambda$ is a transversal slice to Gr_G^μ inside $\overline{\text{Gr}}_G^\lambda$. \square

2.6. Functions on $\mathcal{W}_{G,\mu}$. Let $\mathbb{C}[\overline{\mathcal{W}}_{G,\mu}^\lambda]$ denote the ring of functions on $\overline{\mathcal{W}}_{G,\mu}^\lambda$ and let

$$\mathbb{C}[\mathcal{W}_{G,\mu}] = \varprojlim \mathbb{C}[\overline{\mathcal{W}}_{G,\mu}^\lambda]$$

be the ring of functions on the ind-scheme $\mathcal{W}_{G,\mu}$. The group $T \times \mathbb{C}^*$ acts on $\overline{\mathcal{W}}_{G,\mu}^\lambda$ and $\mathcal{W}_{G,\mu}$ and thus it acts on the corresponding ring of functions.

For any linear algebraic group H , we are going to denote by H_n the subgroup of $H[t^{-1}]$ consisting of those maps $h(t)$ which are equal to the identity $e \in H$ modulo t^{-n} ; in particular, $H_0 = H[t^{-1}]$. Also, let $R_n = \mathbb{C}[t^{-1}]/t^{-n}$; for any scheme X over \mathbb{C} we can consider the scheme of maps $\text{Spec}(R_n) \rightarrow X$ which (abusing slightly the notation) we shall denote by $X(R_n)$. Also, given a \mathbb{C} -point $x \in X$ we shall denote by $X(R_n)_{\text{based}}$ the closed sub-scheme of based maps $\text{Spec}(R_n) \rightarrow X$ (i.e. those maps which send the unique \mathbb{C} -point of $\text{Spec}(R_n)$ to x). In particular, if H is an algebraic group over \mathbb{C} then $H_1/H_n = H(R_n)_{\text{based}}$ (where the role of the point x is played by the identity $e \in H$).

Let $\text{St}_\mu \subset G_1 = G[t^{-1}]_1$ be the stabilizer of t^μ in G_1 . Thus $\mathcal{W}_\mu = G[t^{-1}]_1/\text{St}_\mu$.

Lemma 2.7. (1) Fix $n \in \mathbb{Z}_{>0}$ and let $\mu \in \Lambda^+$ satisfy the following condition:

$$\langle \mu, \check{\alpha} \rangle \geq n \text{ for every positive root } \check{\alpha} \text{ of } \mathfrak{g}. \quad (2.2)$$

Then the image of St_μ in $G_1/G_n = G(R_n)_{\text{based}}$ is equal to $U_-(R_n)_{\text{based}}$. In particular, we have a natural map $\pi_{\mu,n} : \mathcal{W}_{G,\mu} \rightarrow G(R_n)_{\text{based}}/U_-(R_n)_{\text{based}}$.

- (2) Assume that condition (2.2) is satisfied. Then for every $k < n$ the map $\pi_{\mu,n}^* : \mathbb{C}[G(R_n)_{\text{based}}/U_-(R_n)_{\text{based}}] \rightarrow \mathbb{C}[\mathcal{W}_{G,\mu}]$ induces an isomorphism on functions of homogeneity degree k with respect to \mathbb{C}^* .

Proof. (1) is obvious, so let us prove (2). First, let us discuss some preliminary facts about the algebra $\mathbb{C}[G_1]$. It is clear any regular function $F : G \rightarrow \mathbb{C}$ defines a map of ind-schemes $F_1 : G_1 \rightarrow \mathbb{C}[t^{-1}]$ such that for any $g(t) \in G_1$ the free term of $F_1(g(t))$ equal to $F(e)$. Thus for any $i > 0$ we can define the function $a_{F,i}$ on G_1 as the coefficient of t^{-i} in F_1 . It is easy to see that the algebra $\mathbb{C}[G_1]$ is topologically generated by all the $a_{F,i}$. Since every $a_{F,i}$ has degree i with respect to \mathbb{C}^* , it follows that any function of homogeneity degree $< n$ lies in the subalgebra generated by $a_{F,i}$ with $i < n$. On the other hand, if $i < n$ then any $a_{F,i}$ is invariant under the (normal) subgroup G_n of G_1 . Hence, any function on G_1 of homogeneity degree $< n$ is invariant under G_n .

Let f be a function on $\mathcal{W}_{G,\mu}$ of homogeneity degree k with respect to \mathbb{C}^* . Then we can think of f as a function on G_1 which is invariant on the right under St_μ . Then the above discussion shows that f is automatically (left and right) invariant under G_n . In addition, since f is invariant under St_μ it follows from (1) that (under the condition (2.2)) the function f comes from a function \bar{f} on $G_1/G_n \cdot \text{St}_\mu = G(R_n)_{\text{based}}/U_-(R_n)_{\text{based}}$. \square

Now we pass to the main technical result of this Section. Let $\alpha \in \Lambda_+$. Then we have a natural map $Z_{\mathfrak{g}}^\alpha \rightarrow G_1/U_{-,1}$. This map is defined as follows: let (\mathcal{F}_T, κ) be a quasi-map in $Z_{\mathfrak{g}}^\alpha$. The fiber of \mathcal{F}_T at ∞ is automatically trivialized. This trivialization uniquely extends to $\mathbf{C} - \{0\}$ and thus we get a based map $\mathbf{C} - \{0\} \rightarrow \overline{G/U_-}$. Restricting this map to n -th infinitesimal neighbourhood of ∞ in \mathbf{C} we get a natural morphism $Z_{\mathfrak{g}}^\alpha \rightarrow G(R_n)_{\text{based}}/U_-(R_n)_{\text{based}}$.

Theorem 2.8. (1) Let $\lambda, \mu \in \Lambda^+$ such that $\lambda \geq \mu$ and let $\alpha = \lambda - \mu$. Then there exists a natural birational $T \times \mathbb{C}^*$ -equivariant morphism $s_\mu^\lambda : \overline{W}_{G,\mu}^\lambda \rightarrow Z_{\mathfrak{g}}^\alpha$, such that for any n satisfying (2.2), the following diagram is commutative

$$\begin{array}{ccc} \overline{W}_{G,\mu}^\lambda & \xrightarrow{s_\mu^\lambda} & Z_{\mathfrak{g}}^\alpha \\ \pi_{\mu,n} \downarrow & & \downarrow \\ G(R_n)_{\text{based}}/U_-(R_n)_{\text{based}} & \xrightarrow{id} & G(R_n)_{\text{based}}/U_-(R_n)_{\text{based}} \end{array} \quad (2.3)$$

(here we use the right vertical map is described above).

(2) Assume again that (2.2) is satisfied. Then the map $(s_\mu^\lambda)^* : \mathbb{C}[Z_{\mathfrak{g}}^\alpha] \rightarrow \mathbb{C}[\overline{W}_{G,\mu}^\lambda]$ induces an isomorphism on functions of degree $< n$.

Proof. First of all, we claim that (1) implies (2). Indeed, since s_μ^λ is birational, it follows that $(s_\mu^\lambda)^*$ is injective. On the other hand, it is surjective by Lemma 2.7(2) in view of (2.3).

Hence it is enough to explain the construction of s_μ^λ . We start with a modular description of \overline{W}_μ^λ . Recall that Gr_G is the ind-scheme parameterizing a G -bundle \mathcal{F}_G on \mathbf{C} together with a trivialization on $\mathbf{C} - \{0\}$. Also, the isomorphism classes of G -bundles on \mathbf{C} are in one-to-one correspondence with $\Lambda^+ = \Lambda/W$. This identification can be described as follows: it is obvious that T -bundles on \mathbf{C} are in one-to-one correspondence with elements of Λ . On the other hand, it is well-known that any G -bundle on \mathbf{C} has a reduction to T . Thus we get a surjective map from Λ to isomorphism classes of G -bundles on \mathbf{C} and it is easy to see that two T -bundles on \mathbf{C} are isomorphic as G -bundles if and only if one is obtained from the other by means of twist by some $w \in W$. The $G[t^{-1}]$ -orbit $W_{G,\mu} \subset \text{Gr}_G$ (see Section 2.4) parameterizes the G -bundles of isomorphism type $W\mu$ equipped with a trivialization on $\mathbf{C} - \{0\}$. According to Lemma 2.5 we have a contraction $c : W_{G,\mu} \rightarrow G \cdot t^\mu$, and $W_{G,\mu} = c^{-1}(t^\mu)$. It remains to describe the contraction c to the partial flag variety $G \cdot t^\mu$ in modular terms.

Let $HN(\mathcal{F}_G)$ be the Harder-Narasimhan flag of a G -bundle $\mathcal{F}_G \in W_{G,\mu}$. Since \mathcal{F}_G is trivialized off $0 \in \mathbf{C}$, the fiber of $HN(\mathcal{F}_G)$ at $\infty \in \mathbf{C}$ lies in the partial flag variety $G \cdot t^\mu$. So the value of $c(\mathcal{F}_G)$ is just the fiber of the Harder-Narasimhan flag of \mathcal{F}_G at $\infty \in \mathbf{C}$. All in all, $\overline{W}_\mu^\lambda \subset \text{Gr}_G$ parameterizes G -bundles on \mathbf{C} equipped with a trivialization off $0 \in \mathbf{C}$ with a pole of order $\leq \lambda$ at 0 , such that the isomorphism class of \mathcal{F}_G is $W\mu$, and the fiber of the Harder-Narasimhan flag of \mathcal{F}_G at $\infty \in \mathbf{C}$ is the base point $t^\mu \in G \cdot t^\mu$.

Now let us view the Harder-Narasimhan flag of $\mathcal{F}_G \in \overline{W}_{G,\mu}^\lambda$ as a reduction \mathcal{F}_P of \mathcal{F}_G to a parabolic subgroup $P \subset G$ (the stabilizer of t^μ , containing B_-). Let L be the Levi quotient of P , and let L' be the quotient of L modulo center. Then $\text{Ind}_P^{L'} \mathcal{F}_P$ is trivial. Hence the standard complete flag (with stabilizer B_-) in the fiber of \mathcal{F}_G at $\infty \in \mathbf{C}$ canonically extends to the complete flag in $\text{Ind}_P^{L'} \mathcal{F}_P$. Thus any $\mathcal{F}_G \in \overline{W}_{G,\mu}^\lambda$ is canonically equipped with a complete flag \varkappa (a reduction to the Borel) with the standard fiber e_- at $\infty \in \mathbf{C}$.

Finally, we are ready for the construction of s_μ^λ . Given $\mathcal{F}_G \in \overline{W}_{G,\mu}^\lambda$ equipped with an isomorphism $\sigma : \mathcal{F}_G \xrightarrow{\sim} \mathcal{F}_G^{\text{triv}}$ defined off $0 \in \mathbf{C}$, we transfer the canonical complete flag \varkappa to $\mathcal{F}_G^{\text{triv}}$ to obtain a B -structure $\sigma(\varkappa)$ on $\mathcal{F}_G^{\text{triv}}$ with a singularity at $0 \in \mathbf{C}$. Twisting by $\lambda \cdot 0$

we obtain a (regular) generalized B -structure κ on $\mathcal{F}_G^{\text{triv}}$ with an *a priori* defect at $0 \in \mathbf{C}$ (cf. [17, Section 11]). Clearly κ has no defect off $0 \in \mathbf{C}$, its value at $\infty \in \mathbf{C}$ is e_- , and the degree of κ equals $\alpha = \lambda - \mu$. We define $s_\mu^\lambda(\mathcal{F}_G, \sigma) = \kappa$. It follows from *loc. cit.* that s_μ^λ maps $\mathcal{S}^\lambda \cap \overline{\mathcal{W}}_{G,\mu}^\lambda$ isomorphically onto $\overset{\circ}{Z}_g^\alpha$. Here \mathcal{S}^λ is the semi-infinite orbit $U((t)) \cdot t^\lambda$.

The theorem is proved. \square

Corollary 2.9. Z_g^α is normal.

Proof. $\overline{\text{Gr}}_G^\lambda$ is normal (see [12]); hence $\overline{\mathcal{W}}_{G,\mu}^\lambda$ is normal by Lemma 2.5(3). Suppose a function $f \in \mathbb{C}(Z_g^\alpha)$ is a root of a unitary polynomial $f^r + a_{r-1}f^{r-1} + \dots + a_0 = 0$ with coefficients in $\mathbb{C}[Z_g^\alpha]$. We choose n bigger than the degrees of the (homogeneous components) of the coefficients a_i , $1 \leq i \leq r$. Now we choose $\mu \in \Lambda^+$ satisfying (2.2) and such that $\lambda = \mu + \alpha \in \Lambda^+$. Then all the coefficients a_i lie in $\mathbb{C}[\overline{\mathcal{W}}_{G,\mu}^\lambda]$. Hence $f \in \mathbb{C}(Z_g^\alpha) = \mathbb{C}(\overline{\mathcal{W}}_{G,\mu}^\lambda)$ lies in $\mathbb{C}[\overline{\mathcal{W}}_{G,\mu}^\lambda]$. Moreover, the degree of (the highest homogeneous component of) f is less than n . Hence $f \in \mathbb{C}[Z_g^\alpha]$. \square

3. NORMALITY OF AFFINE ZASTAVA FOR $G = \text{SL}(N)$

The purpose of this Section is to give another prove of normality of Zastava for $G = \text{SL}(N)$ which works also in the affine case.

3.1. Notations. We denote by I the set of simple coroots of the affine group $G_{\text{aff}} = \text{SL}(N)_{\text{aff}}$. For $\alpha \in \mathbb{N}[I]$ we denote by Z^α the Drinfeld Zastava space. In [18], [7] we have constructed a normal scheme \mathfrak{Z}^α together with a morphism $\eta : \mathfrak{Z}^\alpha \rightarrow Z^\alpha$ giving a bijection at the level of \mathbb{C} -points. In this section we prove that η is an isomorphism.

Recall that \mathfrak{Z}^α is defined as the categorical quotient $\text{M}^\alpha // G_\alpha$ where M^α is the moduli scheme of representations of a certain chainsaw quiver with relations Q of dimension α . According to [7, 2.3–2.5], the stacky quotient $\text{M}^\alpha / G_\alpha$ is the moduli stack $\text{Perv}^\alpha(\mathcal{S}_N, \mathcal{D}_\infty)$ of perverse coherent sheaves on the Deligne-Mumford stack \mathcal{S}_N equipped with a framing at the divisor $\mathcal{D}_\infty \subset \mathcal{S}_N$. Let us denote by $\mathfrak{g}_3 : \text{Perv}^\alpha(\mathcal{S}_N, \mathcal{D}_\infty) \rightarrow \mathfrak{Z}^\alpha$ the canonical map, and let us denote by $\mathfrak{g}_Z : \text{Perv}^\alpha(\mathcal{S}_N, \mathcal{D}_\infty) \rightarrow Z^\alpha$ the composition of \mathfrak{g}_3 with η . Let us denote by $z_0^Z \in Z^\alpha$ (resp. $z_0^3 \in \mathfrak{Z}^\alpha$) the unique point fixed by the loop rotation action of \mathbb{G}_m . According to [5, 5.14], in order to prove that η is an isomorphism over the base field \mathbb{C} , it suffices to check that the inclusion $\mathfrak{g}_3^{-1}(z_0^3) \hookrightarrow \mathfrak{g}_Z^{-1}(z_0^Z)$ is an equality. We will do this mimicking the argument of [5, 5.16–5.17].

3.2. Perverse sheaves framed off the origin. We consider the following closed substack $\text{Perv}^\alpha(\mathcal{S}_N, \mathcal{S}_N - 0)$: for a scheme S , its S -points are S -families of coherent perverse sheaves on \mathcal{S}_N of degree α equipped with a framing on $\mathcal{S}_N - 0$, i.e. an isomorphism $\mathcal{F}|_{\mathcal{S}_N - 0} \simeq \mathcal{O}_{\mathcal{S}_N} \oplus \mathcal{O}_{\mathcal{S}_N}(-\mathcal{D}_0) \oplus \dots \oplus \mathcal{O}_{\mathcal{S}_N}((1-N)\mathcal{D}_0)$, and satisfying the following condition: for any choice of presentation $h_c : \mathcal{S}_N = \mathbf{C} \times \mathcal{X}_N$ and the corresponding factorization map $\pi_c : Z^\alpha \rightarrow (\mathbf{C} - \infty_c)^\alpha$ the composition $S \rightarrow \text{Perv}^\alpha(\mathcal{S}_N, \mathcal{D}_\infty) \rightarrow Z^\alpha \rightarrow (\mathbf{C} - \infty_c)^\alpha$ sends S to the point $\alpha \cdot 0_{\mathbf{C}}$.

Note that the choices of h_c are parameterized by $c \in \mathbb{A}^1$ according to the choices of vertical directions \mathbf{d}_v in [5, 5.14]. Similarly to [5, Proposition 5.16] we have

Lemma 3.3. *The composition $\mathrm{Perv}^\alpha(\mathcal{S}_N, \mathcal{S}_N - 0) \rightarrow \mathrm{Perv}^\alpha(\mathcal{S}_N, \mathcal{D}_\infty) \rightarrow \mathfrak{Z}^\alpha$ is the constant map to the point $z_0^3 \in \mathfrak{Z}^\alpha$.*

Proof. For a collection $(A_l, B_l, p_l, q_l)_{l \in \mathbb{Z}/N\mathbb{Z}}$ representing a point of $\mathrm{Perv}^\alpha(\mathcal{S}_N, \mathcal{D}_\infty)$, let us denote by T_{W_l} any endomorphism of the line W_l obtained by composing the maps A_k, B_i, p_j, q_r , and by T_{V_l} any similarly obtained endomorphism of V_l . It is well known that the ring of regular functions on \mathfrak{Z}^α is generated by all the possible T_{W_l} 's and the traces of all the possible T_{V_l} 's.

Let \mathcal{F} be an S -point of $\mathrm{Perv}^\alpha(\mathcal{S}_N, \mathcal{S}_N - 0)$. For an integer m , let \mathcal{F}' be the constant S -family of coherent perverse sheaves on \mathcal{S}_N corresponding to the torsion-free sheaf $\mathfrak{m}_0^m \oplus \mathfrak{m}_0^m(-\mathcal{D}_0) \oplus \dots \oplus \mathfrak{m}_0^m((1-N)\mathcal{D}_0)$ where \mathfrak{m}_0 is the maximal ideal of the point $0 \in \mathcal{S}_N$. Then, when m is large enough, we can find a map $\mathcal{F}' \rightarrow \mathcal{F}$ respecting the framings of both sheaves on $\mathcal{S}_N - 0$. The cone of this map is set-theoretically supported at $0 \in \mathcal{S}_N$ and has cohomology in degrees 0, 1.

Let $(V_l, W_l, A_l, B_l, p_l, q_l)_{l \in \mathbb{Z}/N\mathbb{Z}}$ (resp. $(V'_l, W'_l, A'_l, B'_l, p'_l, q'_l)_{l \in \mathbb{Z}/N\mathbb{Z}}$) be the linear algebra data corresponding to \mathcal{F} (resp. \mathcal{F}'). From the constructions of [7, 2.3–2.5] it follows that there are maps $V'_l \rightarrow V_l$, $W'_l \xrightarrow{\sim} W_l$ which commute with all the homomorphisms. Moreover, $q'_l \equiv 0$. From this we obtain that all the T_{W_l} 's vanish, and the only nonzero T_{V_l} 's are matrices of the form $A_{l_1}^{k_1} \circ B^{k_2} \circ A_{l_3}^{k_3} \circ \dots \circ B^{k_{M-1}} \circ A_{l_M}^{k_M}$ where $l_1 = l_M = l$, and $B^{k_{2i}}$ stands for the composition of k_{2i} successive (composable) matrices of the form B_r , and $l_{2i+1} + k_{2i} = l_{2i-1}$. It remains to show that any such matrix is traceless.

As we already noted, for any matrix T'_{V_l} defined as in the previous paragraph but with certain B_r replaced by $p_{r+1}q_r$, the trace vanishes (due to the cyclic invariance of the trace of a product, being equal to the trace of the corresponding endomorphism $T'_{V'_l} = 0$). Using the relation $A_{r+1}B_r - B_rA_r + p_{r+1}q_r = 0$ repeatedly we see that $\mathrm{Tr} T_{V_l} = \mathrm{Tr}(A_l^k(B_{l-1} \circ B_{l-2} \circ \dots \circ B_l)^{k'}) = \mathrm{Tr}((B_{l-1} \circ B_{l-2} \circ \dots \circ B_l)^{k'} A_l^k)$ for certain k, k' . Therefore, it suffices to show that the characteristic polynomial of a matrix $A_l + cB_{l-1} \circ B_{l-2} \circ \dots \circ B_l$ equals $t^{\dim V_l}$ for all $c \in \mathbb{C}$. However, this characteristic polynomial is nothing but the value at our point $(A_l, B_l, p_l, q_l)_{l \in \mathbb{Z}/N\mathbb{Z}} \in \mathbf{M}^\alpha$ of the l -th component of the factorization map $\mathbf{M}^\alpha \rightarrow \mathfrak{Z}^\alpha \rightarrow Z^\alpha \xrightarrow{\pi_c} \mathbb{A}^\alpha \rightarrow \mathbb{A}^{(a_l)}$ (here a_l is the l -th component of $\alpha \in \mathbb{N}[I]$). This completes the proof of the lemma. \square

Lemma 3.4. *For a scheme S , any S -point of the stack $\mathbf{g}_Z^{-1}(z_0^Z)$ factors through an S -point of $\mathrm{Perv}^\alpha(\mathcal{S}_N, \mathcal{S}_N - 0)$.*

Proof. Repeats the argument of [5, 5.17]. One has only to replace the word “trivialization” in *loc. cit.* by “framing”, and \mathbf{S} by \mathcal{S}_N . \square

Now an application of [5, Lemma 5.15] establishes

Theorem 3.5. *Over the base field \mathbb{C} , the morphism $\eta : \mathfrak{Z}^\alpha \rightarrow Z^\alpha$ is an isomorphism.*

Corollary 3.6. *Over the base field \mathbb{C} , the Zastava scheme Z^α is reduced, normal, Gorenstein, and has rational singularities.*

Proof. \mathfrak{Z}^α is proved to be reduced and normal in [18, Theorem 2.7]. Recall the resolution $\varpi : \mathcal{P}^\alpha \rightarrow Z^\alpha$ by the affine Laumon space (see e.g. *loc. cit.*). We will prove that \mathcal{P}^α is

Calabi-Yau. Then it follows by the Grauert-Riemenschneider Theorem that Z^α has rational singularities and is Gorenstein. In order to prove that \mathcal{P}^α is Calabi-Yau, note that if the support of α is not the whole of $I = \mathbb{Z}/N\mathbb{Z}$, then we are in the finite (as opposed to affine) situation, and the Calabi-Yau property of Laumon resolution is proved in [19, Theorem 3] (cf. [15, Corollary 4.3]). If the support of α is full, recall the boundary divisor $\partial Z^\alpha \subset Z^\alpha$ introduced in [5, 11.8]. The proof of [5, Theorem 11.9] shows that for an integer $M \in \mathbb{N}$ the divisor $M\partial Z^\alpha$ (i.e. all the components $\partial_l Z^\alpha$, $l \in I$, of the boundary enter with the same multiplicity M) is a principal divisor. Let us denote $\eta^{-1}(\partial_l Z^\alpha)$ by $\partial_l \mathcal{P}^\alpha$. We see that $M \sum_{l \in I} \partial_l \mathcal{P}^\alpha$ is a principal divisor in \mathcal{P}^α .

Now recall the meromorphic symplectic form Ω on \mathcal{P}^α (see [18, 3.1–3.2]). Let $\omega = \Lambda^{\text{top}} \Omega$ be the corresponding meromorphic volume form on \mathcal{P}^α . The calculation of [18, Proposition 3.5] shows that the divisor of poles of ω equals $\sum_{l \in I} \partial_l \mathcal{P}^\alpha$. We conclude that the canonical class of \mathcal{P}^α is torsion. However, the Picard group of \mathcal{P}^α has no torsion since \mathcal{P}^α is cellular. This completes the proof of the corollary. \square

Corollary 3.7. *The resolution by the affine Laumon space $\varpi : \mathcal{P}^\alpha \rightarrow Z^\alpha$ induces an isomorphism $\varpi^* : \Gamma(Z^\alpha, \mathcal{O}_{Z^\alpha}) \xrightarrow{\sim} \Gamma(\mathcal{P}^\alpha, \mathcal{O}_{\mathcal{P}^\alpha})$. The higher cohomology of the structure sheaf of the affine Laumon space vanishes: $H^k(\mathcal{P}^\alpha, \mathcal{O}_{\mathcal{P}^\alpha}) = 0$ for $k > 0$.*

4. THE BOUNDARY OF ZASTAVA

4.1. The Cartier property. In this section \mathfrak{g} will be an arbitrary finite dimensional simple or untwisted affine Lie algebra with coroot lattice $\Lambda_{\mathfrak{g}}$. Let $\Lambda_{\mathfrak{g}}^+ \subset \Lambda_{\mathfrak{g}}$ denote the cone of positive linear combinations of positive simple coroots. Let T be the torus with the cocharacter lattice $\Lambda_{\mathfrak{g}}$. For $\alpha \in \Lambda_{\mathfrak{g}}^+$, the Zastava scheme $Z_{\mathfrak{g}}^\alpha$ is constructed in [5, Section 9] (under the name of $\mathfrak{U}_{G,B}^\alpha$). It is a certain closure of the (smooth) scheme $\overset{\circ}{Z}_{\mathfrak{g}}^\alpha$ of degree α based maps from (\mathbb{P}^1, ∞) to the Kashiwara flag scheme $\mathcal{B}_{\mathfrak{g}}$ of \mathfrak{g} . In case $\mathfrak{g} = \mathfrak{sl}(N)_{\text{aff}}$ we have $Z_{\mathfrak{g}}^\alpha = Z^\alpha$ of Section 3. The complement $\partial Z_{\mathfrak{g}}^\alpha := Z_{\mathfrak{g}}^\alpha - \overset{\circ}{Z}_{\mathfrak{g}}^\alpha$ (the boundary) is a quasi-effective Cartier divisor in $Z_{\mathfrak{g}}^\alpha$ according to [5, Theorem 11.9]. More precisely, there is a rational function F_α on $Z_{\mathfrak{g}}^\alpha$ whose lift to the normalization of $Z_{\mathfrak{g}}^\alpha$ is regular and has the preimage of $\partial Z_{\mathfrak{g}}^\alpha$ as the zero-divisor.

In general, the boundary $\partial Z_{\mathfrak{g}}^\alpha$ is not irreducible; its irreducible components $\partial_{\alpha_i} Z_{\mathfrak{g}}^\alpha$ are numbered by the simple coroots α_i which enter α with a nonzero coefficient. The argument in [5, 11.5–11.7] gives the order of vanishing of F_α at the generic point of $\partial_{\alpha_i} Z_{\mathfrak{g}}^\alpha$. To formulate the answer we assume that *all* the simple coroots enter α with nonzero coefficients (i.e. α has the full support); otherwise the question reduces to the similar one for a Levi subalgebra of \mathfrak{g} .

Lemma 4.2. *If α_i is a short coroot, the order of vanishing of F_α at the generic point of $\partial_{\alpha_i} Z_{\mathfrak{g}}^\alpha$ is 1; if α_i is a long coroot, the order of vanishing of F_α at the generic point of $\partial_{\alpha_i} Z_{\mathfrak{g}}^\alpha$ is the square length ratio of a long and a short coroot (that is, 1, 2 or 3).* \square

4.3. The degree of F_α . The function F_α is an eigenfunction of the torus $T \times \mathbb{G}_m^*$. Here T (the Cartan torus) acts on $Z_{\mathfrak{g}}^\alpha$ via the change of framing at infinity, and \mathbb{G}_m^* (loop rotations) acts on the source (\mathbb{P}^1, ∞) , and hence on $Z_{\mathfrak{g}}^\alpha$ via the transport of structure. We denote the coordinates on $T \times \mathbb{G}_m^*$ by (z, q) . We define an isomorphism $\alpha \mapsto \alpha^*$ from the coroot lattice

of (G, T) to the root lattice of (G, T) in the basis of simple coroots as follows: $\alpha_i^* := \check{\alpha}_i$ (the corresponding simple root). For an element α of the coroot lattice of (G, T) we denote by z^{α^*} the corresponding character of T .

Proposition 4.4. *The eigencharacter of F_α is $q^{(\alpha, \alpha)/2} z^{\alpha^*}$.*

The proposition will be proved in Section 4.9.

4.5. Deligne pairing. In order to compute the eigencharacter of F_α we recall the construction of F_α following Faltings [12]. To this end recall that given a family $f : \mathcal{X} \rightarrow S$ of smooth projective curves and two line bundles \mathcal{L}_1 and \mathcal{L}_2 on \mathcal{X} Deligne defines a line bundle $\langle \mathcal{L}_1, \mathcal{L}_2 \rangle$ on S . In terms of determinant bundles the definition is simply

$$\langle \mathcal{L}_1, \mathcal{L}_2 \rangle = \det Rf_*(\mathcal{L}_1 \otimes \mathcal{L}_2) \otimes \det Rf_*(\mathcal{O}_{\mathcal{X}}) \otimes (\det Rf_*(\mathcal{L}_1) \otimes \det Rf_*(\mathcal{L}_2))^{-1}. \quad (4.1)$$

One of the main results of Deligne is that the resulting pairing $\text{Pic}(\mathcal{X}) \times \text{Pic}(\mathcal{X}) \rightarrow \text{Pic}(S)$ is symmetric (obvious) and bilinear (not obvious).

4.6. Determinant bundles on Bun_T . Let T be a torus and let Λ (resp. Λ^\vee) be its coweight (resp. weight) lattice. Let also (\cdot, \cdot) be an even pairing on Λ .

Let also C be a smooth projective curve (say, over a field \mathbb{C}) and let Bun_T denote the moduli stack of T -bundles on C . Then to the above data one associates a line bundle \mathcal{D} on Bun_T in the following way. Let e_1, \dots, e_n be a basis of Λ and let f_1, \dots, f_n be the dual basis. For every $i = 1, \dots, n$ let \mathcal{L}_i denote the corresponding line bundle on $\text{Bun}_T \times C$. Let also $a_{ij} = (e_i, e_j) \in \mathbb{Z}$. Then we define

$$\mathcal{D} = \left(\bigotimes_{i=1}^n \langle \mathcal{L}_i, \mathcal{L}_i \rangle^{\otimes \frac{a_{ii}}{2}} \right) \otimes \left(\bigotimes_{1 \leq i < j \leq n} \langle \mathcal{L}_i, \mathcal{L}_j \rangle^{\otimes a_{ij}} \right). \quad (4.2)$$

It is easy to see that \mathcal{D} does not depend on the choice of the basis (here, of course, we have to use the statement that Deligne's pairing is bilinear).

4.7. The case $C = \mathbf{C}$. Let now $C = \mathbf{C}$ (the projective line). In this case let us denote by ${}'\text{Bun}(T)$ the space of T -bundles trivialized at ∞ . This is a scheme isomorphic to $\Lambda \times \text{Spec}(\mathbb{C})$. We shall denote the pull-back of \mathcal{D} to ${}'\text{Bun}_T$ also by ${}'\mathcal{D}$.

Recall that \mathbb{G}_m acts on \mathbf{C} . If v stands for a coordinate on \mathbb{G}_m , and c stands for a coordinate on $\mathbf{C} = \mathbb{P}^1$, we need the action $v(c) := v^2 c$. Note that for this action any line bundle on \mathbf{C} can be equipped with a \mathbb{G}_m -equivariant structure. Comparing to the action of Section 4.3 we have $q = v^2$. The \mathbb{G}_m -action extends to the action on ${}'\text{Bun}_T$ and Bun_T . Since the construction of \mathcal{D} is completely natural, it follows that ${}'\mathcal{D}$ is \mathbb{G}_m -equivariant. Since every component of ${}'\text{Bun}_T$ is a point, this equivariance is given by a character of \mathbb{G}_m (i.e. an integer) for every $\gamma \in \Lambda$.

Lemma 4.8. *The above integer is equal to (γ, γ) .*

Proof. The main observation is the following. Let \mathcal{L} be a line bundle on \mathbb{P}^1 of degree n . It is isomorphic to $\mathcal{O}(n)$ and therefore it has a unique \mathbb{G}_m -equivariant structure such that the action of \mathbb{G}_m on the fiber at ∞ is trivial (this makes sense since ∞ is a fixed point of \mathbb{G}_m). Then we claim that with respect to this equivariant structure \mathbb{G}_m acts on $\det R\Gamma(\mathcal{L})$ by the character $v \mapsto v^{n(n+1)}$. Indeed if $n \geq 0$ then $H^1(\mathcal{L}) = 0$ and $H^0(\mathcal{L})$ has dimension $n + 1$

with weights $0, 2, \dots, 2n$ and their sum is $n(n+1)$. If $n < 0$ then $H^0(\mathcal{L}) = 0$ and $H^1(\mathcal{L})$ has dimension $-n-1$ with weights $-2, -4, \dots, -2-2(-n-2)$ and their sum is equal to $-2(-n-1) - (-n-1)(-n-2) = -(-n)(-n-1) = -n(n+1)$.

Let now \mathcal{L}_1 and \mathcal{L}_2 be two line bundles on \mathbb{P}^1 of degrees n_1 and n_2 . Then the action of \mathbb{G}_m on $\langle \mathcal{L}_1, \mathcal{L}_2 \rangle$ with respect to the above \mathbb{G}_m -equivariant structure on \mathcal{L}_1 and \mathcal{L}_2 corresponds to the integer

$$(n_1 + n_2 + 1)(n_1 + n_2) - (n_1 + 1)n_1 - (n_2 + 1)n_2 = 2n_1n_2. \quad (4.3)$$

Let now \mathcal{F} be a T -bundle of degree γ and let \mathcal{L}_i be the line bundle associated with $f_i \in \Lambda^\vee$. Then the degree of \mathcal{L}_i is $n_i = f_i(\gamma)$. Note that $\gamma = \sum n_i e_i$ and hence

$$(\gamma, \gamma) = \sum_{i=1}^n a_{ii} n_i^2 + \sum_{1 \leq i < j \leq n} 2a_{ij} n_i n_j.$$

Therefore according to (4.3) the action of \mathbb{G}_m on the fiber of $'\mathcal{D}$ at \mathcal{F} is by the character $v \mapsto v^m$ where

$$m = \sum_{i=1}^n \frac{a_{ii}}{2} (2n_i^2) + \sum_{1 \leq i < j \leq n} 2a_{ij} n_i n_j = (\gamma, \gamma).$$

□

4.9. Determinant bundle on Zastava space. In this section \mathfrak{g} is an arbitrary symmetrizable Kac-Moody Lie algebra, and T is the torus with the cocharacter lattice $\Lambda = \Lambda_{\mathfrak{g}}$: the coroot lattice of \mathfrak{g} . The Kashiwara flag scheme of \mathfrak{g} is denoted by $\mathcal{B}_{\mathfrak{g}}$, and $\overset{\circ}{Z}^\gamma$ denotes the space of based maps $(\mathbb{P}^1, \infty) \rightarrow \mathcal{B}_{\mathfrak{g}}$ of degree γ (see [5, Theorem 18.1]). We let (\cdot, \cdot) be the minimal even W -invariant form on Λ . Then we have the natural maps $f_\gamma : \overset{\circ}{Z}^\gamma \rightarrow \text{Bun}_T^\gamma$, $'f_\gamma : \overset{\circ}{Z}^\gamma \rightarrow '\text{Bun}_T^\gamma$. Consider the line bundle $f_\gamma^* \mathcal{D} = 'f_\gamma^*(' \mathcal{D})$ on $\overset{\circ}{Z}^\gamma$.

This line bundle acquires two different trivializations for the following reasons:

1) Since $'\text{Bun}_T^\gamma$ is just one point, the line bundle \mathcal{D} is trivial there and hence its pull-back is trivial as well.

2) According to Faltings [12, Section 7], there is a trivialization of the similar bundle on the space of all maps from any smooth projective curve C to $\mathcal{B}_{\mathfrak{g}}$ (Faltings proves this only for finite-dimensional \mathfrak{g} but his proof works word by word for any \mathfrak{g}).

Moreover, the line bundle $f_\gamma^* \mathcal{D}$ is naturally \mathbb{G}_m -equivariant (\mathbb{G}_m acts on everything); both trivializations 1) and 2) are compatible with this structure if in 1) we let \mathbb{G}_m act on the trivial bundle via the character $v \mapsto v^{(\gamma, \gamma)}$ (this follows from Lemma 4.8) and in 2) we let \mathbb{G}_m act trivially on the trivial bundle (this follows from the fact that Faltings' construction is natural with respect to everything). Thus 1) and 2) together give us an invertible function F_γ on $\overset{\circ}{Z}^\gamma$ such that $F_\gamma(v\phi) = v^{(\gamma, \gamma)} F_\gamma(\phi)$, that is $F_\gamma(q\phi) = q^{(\gamma, \gamma)/2} F_\gamma(\phi)$.

This completes the proof of the main part of Proposition 4.4: we have found the eigencharacter of F_α with respect to the loop rotations. It remains to compute the eigencharacter of F_α with respect to the Cartan torus. We must check the following. Given a one-parametric subgroup $\beta : \mathbb{G}_m \rightarrow T$, and a general point $\phi \in Z_{\mathfrak{g}}^\alpha$, the action map $\mathbb{G}_m \rightarrow Z_{\mathfrak{g}}^\alpha$, $c \mapsto \beta(c) \cdot \phi$, extends to the map $a : \mathbb{G}_m \subset \mathbb{A}^1 \rightarrow Z_{\mathfrak{g}}^\alpha$. We must check that the function $F_\alpha \circ a$ on \mathbb{A}^1

has the order of vanishing $\langle \beta, \alpha^* \rangle$ at the origin. By the factorization property of $Z_{\mathfrak{g}}^{\alpha}$ the question reduces to the case $\mathfrak{g} = \mathfrak{sl}(2)$, $\alpha = 1$, which is obvious. \square

5. GORENSTEIN PROPERTY OF ZASTAVA

Suppose \mathfrak{g} is a simply laced simple Lie algebra.

Proposition 5.1. *$Z_{\mathfrak{g}}^{\alpha}$ is a Gorenstein (hence, Cohen-Macaulay) scheme with canonical (hence rational) singularities.*

Proof. We are going to apply Elkik's criterion [10] in order to prove that $Z_{\mathfrak{g}}^{\alpha}$ has rational singularities. To this end we will use the Kontsevich resolution $\pi : M_{\mathfrak{g}}^{\alpha} \rightarrow Z_{\mathfrak{g}}^{\alpha}$ (see [13, Section 8]). We will show that the discrepancy of π is strictly positive, that is the singularities of $Z_{\mathfrak{g}}^{\alpha}$ are canonical, hence rational.

Recall that $\overline{M}_{0,0}(\mathbb{P}^1 \times \mathcal{B}_{\mathfrak{g}}, (1, \alpha))$ is the moduli space of stable maps from curves of genus zero without marked points of degree $(1, \alpha)$ to $\mathbb{P}^1 \times \mathcal{B}_{\mathfrak{g}}$. It is a smooth Deligne-Mumford stack equipped with a birational projection to the space of Drinfeld quasimaps from \mathbb{P}^1 to $\mathcal{B}_{\mathfrak{g}}$. If C is such a curve of genus 0, then it has a distinguished irreducible component C_h (h for horizontal) which maps isomorphically onto \mathbb{P}^1 . Using this isomorphism to identify C_h with \mathbb{P}^1 we obtain the points $0, \infty \in C_h$. Now $M_{\mathfrak{g}}^{\alpha} \subset \overline{M}_{0,0}(\mathbb{P}^1 \times \mathcal{B}_{\mathfrak{g}}, (1, \alpha))$ is the locally closed substack cut out by the open condition that $\infty \in C_h$ is a smooth point of C , and by the closed condition that the stable map $\phi : C \rightarrow \mathbb{P}^1 \times \mathcal{B}_{\mathfrak{g}}$ takes $\infty \in C_h \subset C$ to the marked point $B_0 \in \mathcal{B}_{\mathfrak{g}}$.

The open substack $\overset{\circ}{M}_{\mathfrak{g}}^{\alpha} \simeq \overset{\circ}{Z}_{\mathfrak{g}}^{\alpha}$ of genuine based maps is formed by the pairs (C, ϕ) such that C is irreducible. The complement is a normal crossing divisor with irreducible components D_{β} numbered by all $\beta \leq \alpha$. The generic point of D_{β} parameterizes the pairs (C, ϕ) such that $C = C_h \cup C_v$ is a union of 2 irreducible components (v for vertical), and the degree of $\phi|_{C_v}$ equals $(0, \beta)$. If $\beta = \alpha_i$ is a simple root, then $\pi : D_{\alpha_i} \rightarrow Z_{\mathfrak{g}}^{\alpha}$ is a birational isomorphism onto the boundary component $\partial_{\alpha_i} Z_{\mathfrak{g}}^{\alpha} \subset Z_{\mathfrak{g}}^{\alpha}$. If β is not a simple root, then D_{β} is an exceptional divisor of the Kontsevich resolution $\pi : M_{\mathfrak{g}}^{\alpha} \rightarrow Z_{\mathfrak{g}}^{\alpha}$.

Recall the symplectic form Ω on $\overset{\circ}{Z}_{\mathfrak{g}}^{\alpha}$ constructed in [16]. Its top exterior power $\Lambda^{|\alpha|}\Omega$ is a nonvanishing section of the canonical line bundle on $\overset{\circ}{Z}_{\mathfrak{g}}^{\alpha}$. It is well known that the complement $\overset{\bullet}{Z}_{\mathfrak{g}}^{\alpha} \subset Z_{\mathfrak{g}}^{\alpha}$ to the union of codimension at least 2 boundary components is smooth. Let us denote its canonical line bundle by $\overset{\bullet}{\omega}$. Then according to [16, Remark 3], $\Lambda^{|\alpha|}\Omega$ has poles of order 1 at all the boundary divisors $\partial_{\alpha_i} Z_{\mathfrak{g}}^{\alpha} \subset \overset{\bullet}{Z}_{\mathfrak{g}}^{\alpha}$. According to Lemma 4.2, the product $F_{\alpha} \Lambda^{|\alpha|}\Omega$ is a regular nowhere vanishing section of $\overset{\bullet}{\omega}$, hence $\overset{\bullet}{\omega}$ is a trivial line bundle. We conclude that $Z_{\mathfrak{g}}^{\alpha}$ is \mathbb{Q} -Gorenstein with trivial canonical class ω_Z , and the discrepancy of the Kontsevich resolution $\pi : M_{\mathfrak{g}}^{\alpha} \rightarrow Z_{\mathfrak{g}}^{\alpha}$ is isomorphic to its canonical class ω_M . We have $\omega_M \otimes \pi^* \omega_Z^{-1} = \sum_{\beta \leq \alpha} m_{\beta} D_{\beta}$. We know that for $\beta = \alpha_i$ a simple root the multiplicity m_{α_i} is 0, and we will compute the multiplicities for all the rest β , and show that they are all strictly positive. In fact, due to the factorization property of Zastava, it suffices to compute a single multiplicity m_{β} : for $\beta = \alpha$.

Lemma 5.2. $m_{\alpha} = |\alpha| + \frac{(\alpha, \alpha)}{2} - 2$.

Proof. The loop rotations group \mathbb{G}_m (cf. Section 4.3) acts on $M_{\mathfrak{g}}^\alpha$ via its action on the target \mathbb{P}^1 . The fixed point set $D_\alpha^{\mathbb{G}_m}$ contains all the pairs (C, ϕ) such that C consists of 2 irreducible components C_h and C_v intersecting at the point $0 \in C_h$. We will compute m_α via comparison of the \mathbb{G}_m -actions in the fibers of ω_M and the normal bundle \mathcal{N} to D_α at such a fixed point $(C, \phi) \in D_\alpha^{\mathbb{G}_m}$.

The fiber of the normal bundle $\mathcal{N}_{(C, \phi)}$ equals the tensor product of the tangent spaces at 0 to C_h and C_v . Hence \mathbb{G}_m acts on $\mathcal{N}_{(C, \phi)}$ via the character q^{-1} (recall that in the normalization of Section 4.3 and Section 4.7 \mathbb{G}_m acts on the coordinate function on $\mathbb{P}^1 \simeq C_h$ via the character q ; hence it acts on the tangent space at 0 via the character q^{-1}).

According to [22, 1.3], the tangent space $T_{(C, \phi)} D_\alpha$ is $H^1(C, \mathcal{F}^\bullet(-\infty))$ where $\mathcal{F}^\bullet = \mathcal{F}^0 \rightarrow \mathcal{F}^1$ is the following complex of sheaves in degrees 0, 1: \mathcal{F}^0 is the sheaf of vector fields on C vanishing at 0; while $\mathcal{F}^1 = \phi^* \mathcal{T}_{\mathbb{P}^1 \times \mathcal{B}_{\mathfrak{g}}}$ is the pullback of the tangent sheaf of the target. Moreover, according to *loc. cit.*, $H^0(C, \mathcal{F}^\bullet) = H^2(C, \mathcal{F}^\bullet) = 0$.

It is immediate to check that $H^\bullet(C, \mathcal{F}^0(-\infty)) = H^0(C, \mathcal{F}^0(-\infty))$ is a 3-dimensional vector space with \mathbb{G}_m -character $1 + 1 + 1$, hence it contributes to $\det H^1(C, \mathcal{F}^\bullet(-\infty))$ the \mathbb{G}_m -character 1. It is also clear that $H^\bullet(C, \phi^* \mathcal{T}_{\mathbb{P}^1 \times \mathcal{B}_{\mathfrak{g}}}(-\infty)) = H^0(C, \phi^* \mathcal{T}_{\mathbb{P}^1 \times \mathcal{B}_{\mathfrak{g}}}(-\infty))$ is a direct sum of a $2|\alpha|$ -dimensional vector space with trivial \mathbb{G}_m -action, and a 2-dimensional vector space with \mathbb{G}_m -character $1 + q^{-1}$. Hence it contributes the \mathbb{G}_m -character q^{-1} to $\det H^1(C, \mathcal{F}^\bullet(-\infty))$. All in all, \mathbb{G}_m acts on $\det T_{(C, \phi)} M_{\mathfrak{g}}^\alpha$ with the character q^{-2} , and on the fiber of the canonical bundle $\omega_{M, (C, \phi)}$ with the character q^2 .

Now recall that the canonical class ω_Z of $Z_{\mathfrak{g}}^\alpha$ is trivialized by the section $F_\alpha \Lambda^{|\alpha|} \Omega$ which is an eigensection of \mathbb{G}_m with the character $q^{\frac{(\alpha, \alpha)}{2} + |\alpha|}$: this follows from Proposition 4.4 and [16, Remark 3]. Hence \mathbb{G}_m acts on the fiber at (C, ϕ) of the discrepancy line bundle $\omega_M \otimes \pi^* \omega_Z^{-1}$ with the character $q^{2 - \frac{(\alpha, \alpha)}{2} - |\alpha|}$. It coincides with the character of \mathbb{G}_m in the fiber $\mathcal{N}_{(C, \phi)}$ raised to the power $\frac{(\alpha, \alpha)}{2} + |\alpha| - 2$. Hence $\mathcal{O}(D_\alpha)$ must enter $\omega_M \otimes \pi^* \omega_Z^{-1}$ with coefficient $\frac{(\alpha, \alpha)}{2} + |\alpha| - 2$. This completes the proof of the lemma. \square

We return to the proof of the proposition. Since m_α is positive for nonsimple α , the singularities of $Z_{\mathfrak{g}}^\alpha$ are canonical, hence rational, hence Cohen-Macaulay. It remains to prove the Gorenstein property. Let us denote by j the open embedding of $\overset{\bullet}{Z}_{\mathfrak{g}}^\alpha$ into $Z_{\mathfrak{g}}^\alpha$. Let us denote by \mathcal{D}_Z the dualizing sheaf of $Z_{\mathfrak{g}}^\alpha$. We have to check that the natural map $\psi : \mathcal{D}_Z \rightarrow j_* \overset{\bullet}{\omega}$ is an isomorphism (the RHS is a trivial line bundle on $Z_{\mathfrak{g}}^\alpha$). Let $\varpi : Z_{\mathfrak{g}}^\alpha \rightarrow Y$ be a finite map to a smooth affine scheme. Then \mathcal{D}_Z is locally free over Y , and $\varpi_* \psi$ is an isomorphism, hence ψ is an isomorphism itself.

The proposition is proved. \square

6. COHEN-MACAULAY PROPERTY OF THE BOUNDARY OF ZASTAVA

The main purpose of this Section is to show that the boundary components $\partial_\gamma Z_{\mathfrak{g}}^\alpha$ are Cohen-Macaulay in case \mathfrak{g} is simple and simply-laced.

6.1. Structure of the boundary. If $\alpha = \beta + \gamma$ for $\alpha, \beta, \gamma \in \Lambda_{\mathfrak{g}}^+$, then according to [5, Section 10], we have a finite morphism $\iota_{\beta, \gamma} : Z_{\mathfrak{g}}^\beta \times (\mathbf{C} - \infty)^\gamma \rightarrow Z_{\mathfrak{g}}^\alpha$. Its image (a closed

reduced subscheme of $Z_{\mathfrak{g}}^{\alpha}$) will be denoted by $\partial_{\gamma} Z_{\mathfrak{g}}^{\alpha}$. We will denote $Z_{\mathfrak{g}}^{\beta} \times (\mathbf{C} - \infty)^{\gamma}$ (resp. $\partial Z_{\mathfrak{g}}^{\beta} \times (\mathbf{C} - \infty)^{\gamma}$) by $\tilde{\partial}_{\gamma} Z_{\mathfrak{g}}^{\alpha}$ (resp. $\partial \tilde{\partial}_{\gamma} Z_{\mathfrak{g}}^{\alpha}$) for short. The image of $\partial \tilde{\partial}_{\gamma} Z_{\mathfrak{g}}^{\alpha}$ in $\partial_{\gamma} Z_{\mathfrak{g}}^{\alpha}$ (a closed reduced subscheme of $\partial_{\gamma} Z_{\mathfrak{g}}^{\alpha}$) will be denoted by $\partial \partial_{\gamma} Z_{\mathfrak{g}}^{\alpha}$. The union of $\partial_{\gamma} Z_{\mathfrak{g}}^{\alpha}$ over all $\gamma \in \Lambda_{\mathfrak{g}}^{+}$ such that $\alpha - \gamma \in \Lambda_{\mathfrak{g}}^{+}$ and $|\gamma| = n$ (a closed reduced equidimensional subscheme of $Z_{\mathfrak{g}}^{\alpha}$ of codimension n) will be denoted $\partial_n Z_{\mathfrak{g}}^{\alpha}$. Here for $\gamma = \sum c_i \alpha_i$ we set $|\gamma| = \sum c_i$. The disjoint union of $\tilde{\partial}_{\gamma} Z_{\mathfrak{g}}^{\alpha}$ (resp. $\partial \tilde{\partial}_{\gamma} Z_{\mathfrak{g}}^{\alpha}$) over all $\gamma \in \Lambda_{\mathfrak{g}}^{+}$ such that $\alpha - \gamma \in \Lambda_{\mathfrak{g}}^{+}$ and $|\gamma| = n$ will be denoted by $\tilde{\partial}_n Z_{\mathfrak{g}}^{\alpha}$ (resp. $\partial \tilde{\partial}_n Z_{\mathfrak{g}}^{\alpha}$). Thus we have a finite morphism $\iota_n : \tilde{\partial}_n Z_{\mathfrak{g}}^{\alpha} \rightarrow \partial_n Z_{\mathfrak{g}}^{\alpha}$, and the reduced preimage of $\partial_{n+1} Z_{\mathfrak{g}}^{\alpha} \subset \partial_n Z_{\mathfrak{g}}^{\alpha}$ is $\partial \tilde{\partial}_n Z_{\mathfrak{g}}^{\alpha} \subset \tilde{\partial}_n Z_{\mathfrak{g}}^{\alpha}$. Hence we have an embedding $\iota_n^* : \Gamma(\partial_n Z_{\mathfrak{g}}^{\alpha}, \mathcal{O}_{\partial_n Z_{\mathfrak{g}}^{\alpha}}(-\partial_{n+1} Z_{\mathfrak{g}}^{\alpha})) \hookrightarrow \Gamma(\tilde{\partial}_n Z_{\mathfrak{g}}^{\alpha}, \mathcal{O}_{\tilde{\partial}_n Z_{\mathfrak{g}}^{\alpha}}(-\partial \tilde{\partial}_n Z_{\mathfrak{g}}^{\alpha}))$.

6.2. Factorization base change. We have the factorization morphism $\pi : Z_{\mathfrak{g}}^{\alpha} \rightarrow \mathbb{A}^{\alpha}$ to the space of coloured configurations on $\mathbb{A}^1 = \mathbf{C} - \infty$. We also have the space of ordered configurations $\mathbb{A}^{|\alpha|} \rightarrow \mathbb{A}^{\alpha}$. We denote the base change $\mathbb{A}^{|\alpha|} \times_{\mathbb{A}^{\alpha}} Z_{\mathfrak{g}}^{\alpha}$ by $\tilde{Z}_{\mathfrak{g}}^{\alpha} \xrightarrow{\xi} Z_{\mathfrak{g}}^{\alpha}$.

Lemma 6.3. $\tilde{Z}_{\mathfrak{g}}^{\alpha}$ is Cohen-Macaulay.

Proof. Since $Z_{\mathfrak{g}}^{\alpha}$ is Cohen-Macaulay, there is a finite morphism $\eta : Z_{\mathfrak{g}}^{\alpha} \rightarrow X$ such that X is smooth affine, and $\eta_* \mathcal{O}_{Z_{\mathfrak{g}}^{\alpha}}$ is locally free. Since $\mathcal{O}_{\mathbb{A}^{|\alpha|}}$ is free over $\mathcal{O}_{\mathbb{A}^{\alpha}}$, we see that $\mathcal{O}_{\tilde{Z}_{\mathfrak{g}}^{\alpha}}$ is free over $\mathcal{O}_{Z_{\mathfrak{g}}^{\alpha}}$, and hence $(\eta \circ \xi)_* \mathcal{O}_{\tilde{Z}_{\mathfrak{g}}^{\alpha}}$ is locally free over \mathcal{O}_X . We conclude that $\tilde{Z}_{\mathfrak{g}}^{\alpha}$ is Cohen-Macaulay. \square

We denote the base change of a boundary component $\mathbb{A}^{|\alpha|} \times_{\mathbb{A}^{\alpha}} \partial_{\gamma} Z_{\mathfrak{g}}^{\alpha}$ by $\partial_{\gamma} \tilde{Z}_{\mathfrak{g}}^{\alpha}$. Recall that I is the set of simple coroots of \mathfrak{g} . Let us choose a finite set J together with a map $A : J \rightarrow I$ such that $\sharp A^{-1}(i) = a_i$ for any $i \in I$; here $\alpha = \sum_{i \in I} a_i \alpha_i$. In words, (J, A) is an unfolding of α , and $\sharp(J, A) = \alpha$. If A is clear from the context, we will omit it from the notations. For instance, for a subset $K \subset J$ we can restrict A to K , and we have $\sharp K = \beta \leq \alpha$. Clearly, $\partial_{\gamma} \tilde{Z}_{\mathfrak{g}}^{\alpha}$ is a union of irreducible components numbered by the subsets $K \subset J$ such that $\sharp K = \gamma$. Given such $K \subset J$, the corresponding irreducible component $\partial_K \tilde{Z}_{\mathfrak{g}}^{\alpha}$ is isomorphic to $\mathbb{A}^K \times \tilde{Z}_{\mathfrak{g}}^{\alpha - \gamma}$. In particular, $\partial_K \tilde{Z}_{\mathfrak{g}}^{\alpha}$ is Cohen-Macaulay, according to Lemma 6.3. It follows that $\partial_K \tilde{Z}_{\mathfrak{g}}^{\alpha}$ is reduced (being generically reduced).

6.4. Modular interpretation of $\partial_K \tilde{Z}_{\mathfrak{g}}^{\alpha}$. In order to study various unions and intersections of the irreducible components of $\partial_{\gamma} \tilde{Z}_{\mathfrak{g}}^{\alpha}$, we need to describe their modular interpretation. Let $U \subset G$ be the nilpotent radical of a Borel subgroup. The affinization of the base affine space G/U is denoted by $\overline{G/U}$. It is acted upon by the abstract Cartan torus T on the right, and by U on the left. The quotient stack $U \backslash \overline{G/U} / T$ is denoted by \mathcal{Y} . It contains an open substack $\overset{\circ}{\mathcal{Y}}$ which is just a point, corresponding to the open orbit of U on $(G/U)/T$. The complement to this point is a divisor \mathcal{D} with irreducible components \mathcal{D}_i , $i \in I$, each one a Cartier divisor. The complement $\overline{G/U} - G/U$ is a union of irreducible components C_i , $i \in I$, each one of codimension 2. The component C_i gives rise to the closed codimension 2 substack $\mathcal{C}_i \subset \mathcal{Y}$. Note that $\mathcal{C}_i \subset \mathcal{D}_i$.

According to Drinfeld (see e.g. [5]), $Z_{\mathfrak{g}}^{\alpha}$ is the moduli space of degree α based maps from \mathbf{C} to \mathcal{Y} . The factorization morphism $\pi : Z_{\mathfrak{g}}^{\alpha} \rightarrow \mathbb{A}^{\alpha}$ is nothing but the pullback of the

coloured divisor $\mathcal{D} \subset \mathcal{Y}$. Hence $\tilde{Z}_{\mathfrak{g}}^{\alpha}$ represents the functor which associates to a test scheme S the following data: a) a morphism $\phi : S \times \mathbf{C} \rightarrow \mathcal{Y}$ such that $\phi(S \times \infty) \subset \overset{\circ}{\mathcal{Y}}$; b) an S -morphism $\eta : S \times J \rightarrow \phi^* \mathcal{D}$ such that as divisors on $S \times \mathbf{C}$, $\sum_{j \in J} \eta(S \times j) = \phi^* \mathcal{D}$.

The closed subfunctor $\partial_K \tilde{Z}_{\mathfrak{g}}^{\alpha}(S) \subset \tilde{Z}_{\mathfrak{g}}^{\alpha}(S)$ is cut out by the condition $\phi(S \times j) \subset \mathbb{C}_i$ if $j \in K$, and $A(j) = i$.

Since the union (resp. scheme-theoretic intersection) of closed subschemes represents the union (resp. intersection) of the corresponding closed subfunctors, we arrive at the following

Lemma 6.5. *Let $K_0, K_1, \dots, K_m \subset J$ be a collection of subsets. Then the scheme-theoretic intersection $\partial_{K_0} \tilde{Z}_{\mathfrak{g}}^{\alpha} \cap \bigcup_{r=1}^m \partial_{K_r} \tilde{Z}_{\mathfrak{g}}^{\alpha}$ is reduced.* \square

Lemma 6.6. *$\partial_{\gamma} \tilde{Z}_{\mathfrak{g}}^{\alpha}$ is Cohen-Macaulay.*

Proof. Recall that $\partial_{\gamma} \tilde{Z}_{\mathfrak{g}}^{\alpha} = \bigcup_{\#K=\gamma} \partial_K \tilde{Z}_{\mathfrak{g}}^{\alpha}$. Let us choose an ordering of I , and then order J so that $A(j_1) > A(j_2) \implies j_1 > j_2$. To a subset of J we associate the word composed of its elements written in the descending order. Finally, let us order all the subsets of J (i.e. the corresponding words) lexicographically (the smallest j comes first in the lexicographic order). We prove that $\bigcup_{r \leq m} \partial_{K_r} \tilde{Z}_{\mathfrak{g}}^{\alpha}$ is Cohen-Macaulay by induction in m . In effect, the base of induction is already known, and each step follows from Lemma 6.5 and [9, Exercise 18.13]: the union of two Cohen-Macaulay closed subschemes X and Y is Cohen-Macaulay if their (scheme-theoretic) intersection is Cohen-Macaulay and of (pure) codimension 1 in both. \square

Proposition 6.7. *$\partial_{\gamma} Z_{\mathfrak{g}}^{\alpha}$ is Cohen-Macaulay.*

Proof. The algebra of functions $\mathbb{C}[\partial_{\gamma} Z_{\mathfrak{g}}^{\alpha}]$ coincides with the invariants of the finite group \mathfrak{S}_{α} (automorphisms of J over I) in the algebra $\mathbb{C}[\partial_{\gamma} \tilde{Z}_{\mathfrak{g}}^{\alpha}]$. The latter being Cohen-Macaulay by Lemma 6.6, the former inherits the Cohen-Macaulay property. \square

6.8. Seminormality of $\partial_{\gamma} Z_{\mathfrak{g}}^{\alpha}$. Recall that a reduced scheme X over $\text{Spec} \mathbb{C}$ is called seminormal if any homeomorphism from a reduced scheme Y to X is an isomorphism (see e.g. [20]).

Lemma 6.9. *$\partial_{\gamma} Z_{\mathfrak{g}}^{\alpha}$ is seminormal.*

Proof. Being Cohen-Macaulay, $\partial_{\gamma} Z_{\mathfrak{g}}^{\alpha}$ enjoys Serre's S_2 property, so by the criterion [20, Corollary 2.7] it suffices to check that $\partial_{\gamma} Z_{\mathfrak{g}}^{\alpha}$ is seminormal in codimension 1. So we consider a closed point $\phi \in \partial_{\gamma} Z_{\mathfrak{g}}^{\alpha}$ which has defect either γ or $\gamma + \alpha_i$. If the defect is γ , then the point ϕ is smooth. If the defect is $\gamma + \alpha_i$, then the factorization reduces our problem to the case $G = \text{SL}(2)$. Now an explicit calculation shows that étale locally around ϕ , $\partial_{\gamma} Z_{\mathfrak{g}}^{\alpha}$ is isomorphic to the product of a smooth scheme X and a 1-dimensional scheme Y equal to the union of coordinate axes in \mathbb{A}^{c_i+1} where $\gamma = \sum_{i \in I} c_i \alpha_i$. Clearly, Y is seminormal (but not Gorenstein), and we are done. \square

6.10. Functions vanishing on the boundary. Recall the notations of Section 6.1.

Lemma 6.11. *We have an isomorphism*

$$\iota_{|\gamma|}^* : \Gamma(\partial_{\gamma} Z_{\mathfrak{g}}^{\alpha}, \mathcal{O}_{\partial_{\gamma} Z_{\mathfrak{g}}^{\alpha}}(-\partial \partial_{\gamma} Z_{\mathfrak{g}}^{\alpha})) \xrightarrow{\sim} \Gamma(\tilde{\partial}_{\gamma} Z_{\mathfrak{g}}^{\alpha}, \mathcal{O}_{\tilde{\partial}_{\gamma} Z_{\mathfrak{g}}^{\alpha}}(-\partial \tilde{\partial}_{\gamma} Z_{\mathfrak{g}}^{\alpha})).$$

Proof. According to [23, Theorem 2.2], the algebra of regular functions on the seminormal variety $\partial_\gamma Z_{\mathfrak{g}}^\alpha$ coincides with the subalgebra of $\mathbb{C}[\tilde{\partial}_\gamma Z_{\mathfrak{g}}^\alpha]$ formed by all the regular functions f such that $f(y_1) = f(y_2)$ whenever $\iota_{\beta,\gamma}(y_1) = \iota_{\beta,\gamma}(y_2)$. In particular, any function in $\mathbb{C}[\tilde{\partial}_\gamma Z_{\mathfrak{g}}^\alpha]$ vanishing at $\partial\tilde{\partial}_\gamma Z_{\mathfrak{g}}^\alpha$ is the pullback of a function on $\partial_\gamma Z_{\mathfrak{g}}^\alpha$ (automatically vanishing at $\partial\partial_\gamma Z_{\mathfrak{g}}^\alpha$). \square

Lemma 6.12. *We have an isomorphism*

$$\begin{aligned} \iota_n^* : \Gamma(\partial_n Z_{\mathfrak{g}}^\alpha, \mathcal{O}_{\partial_n Z_{\mathfrak{g}}^\alpha}(-\partial\partial_n Z_{\mathfrak{g}}^\alpha)) &\xrightarrow{\sim} \Gamma(\tilde{\partial}_n Z_{\mathfrak{g}}^\alpha, \mathcal{O}_{\tilde{\partial}_n Z_{\mathfrak{g}}^\alpha}(-\partial\tilde{\partial}_n Z_{\mathfrak{g}}^\alpha)) = \\ &= \bigoplus_{\substack{|\gamma|=n \\ \gamma \leq \alpha}} \Gamma(\tilde{\partial}_\gamma Z_{\mathfrak{g}}^\alpha, \mathcal{O}_{\tilde{\partial}_\gamma Z_{\mathfrak{g}}^\alpha}(-\partial\tilde{\partial}_\gamma Z_{\mathfrak{g}}^\alpha)) = \bigoplus_{\substack{|\gamma|=n \\ \gamma \leq \alpha}} \Gamma(\partial_\gamma Z_{\mathfrak{g}}^\alpha, \mathcal{O}_{\partial_\gamma Z_{\mathfrak{g}}^\alpha}(-\partial\partial_\gamma Z_{\mathfrak{g}}^\alpha)). \end{aligned}$$

Proof. We will prove that the natural inclusion

$$\Gamma(\partial_n Z_{\mathfrak{g}}^\alpha, \mathcal{O}_{\partial_n Z_{\mathfrak{g}}^\alpha}(-\partial\partial_n Z_{\mathfrak{g}}^\alpha)) \hookrightarrow \bigoplus_{|\gamma|=n} \Gamma(\partial_\gamma Z_{\mathfrak{g}}^\alpha, \mathcal{O}_{\partial_\gamma Z_{\mathfrak{g}}^\alpha}(-\partial\partial_\gamma Z_{\mathfrak{g}}^\alpha))$$

is an isomorphism. To this end we essentially repeat the proof of Lemma 6.11: First we prove that $\partial_n Z_{\mathfrak{g}}^\alpha = \bigcup_{|\gamma|=n} \partial_\gamma Z_{\mathfrak{g}}^\alpha$ is Cohen-Macaulay by the criterion [9, Exercise 18.13] (noting that the intersection of two boundary components is reduced and equal to a smaller boundary component). Second we prove that $\partial_n Z_{\mathfrak{g}}^\alpha$ is seminormal in codimension 1, and deduce that $\partial_n Z_{\mathfrak{g}}^\alpha$ is seminormal. Third, we conclude that the regular functions on $\partial_n Z_{\mathfrak{g}}^\alpha$ coincide with the regular functions on $\tilde{\partial}_n Z_{\mathfrak{g}}^\alpha$ which agree on the fibers of ι_n . The desired statement follows. \square

7. FERMIONIC FORMULAS AND THE BOUNDARY OF ZASTAVA

7.1. Equivariant K -theory of affine Laumon spaces. We recall some facts from [6] and [14]. We consider the equivariant K -theory of the affine Laumon spaces \mathcal{P}^α with respect to certain torus $\hat{T} = \tilde{T} \times \mathbb{C}^* \times \mathbb{C}^*$. Here \tilde{T} is a certain 2^{N-1} -fold cover of a Cartan torus $T \subset \mathrm{SL}(N)$ acting on \mathcal{P}^α via the change of framing at infinity, while $\mathbb{C}^* \times \mathbb{C}^*$ acts on \mathbb{S}_N by dilations, and hence on \mathcal{P}^α by the transport of structure. The coordinates on \hat{T} are denoted by (t_1, \dots, t_N, u, v) , $t_1 \cdots t_N = 1$. Certain natural correspondences between the affine Laumon spaces give rise to the action of the affine quantum group \mathcal{U} of type \tilde{A}_{N-1} (a $\mathbb{C}[v^{\pm 1}]$ -algebra) on $\mathcal{M} = \bigoplus_\alpha \mathcal{M}_\alpha := \bigoplus_\alpha K^{\hat{T}}(\mathcal{P}^\alpha) \otimes_{K^{\hat{T}}(pt)} \mathrm{Frac}(K^{\hat{T}}(pt))$. The Cartan subalgebra of \mathcal{U} is $\mathcal{U}_0 = \mathbb{C}[v^{\pm 1}, C^{\pm 1}, L_i^{\pm 1}, 0 \leq i \leq N-1]$ in notations of [6, 3.5]. Now \mathcal{M} contains the universal Verma module \mathcal{M}' over $\mathcal{U}' = \mathcal{U} \otimes \mathrm{Frac}(\mathcal{U}_0)$; here C acts as uv^N , while for $\alpha = \sum_{i=0}^{N-1} d_i \alpha_i$ the action of L_i on \mathcal{M}_α is given by $t_1^{-1} \cdots t_i^{-1} v^{d_i + i(n-i)/2}$, see [6, 3.4, 3.17]. A certain completion $\hat{\mathcal{M}}'$ contains the Whittaker vector $\mathbf{u} = \sum_\alpha \mathbf{u}_\alpha$, and the dual Whittaker vector $\mathbf{n}' = \sum_\alpha \mathbf{n}'_\alpha$. According to [6, Corollary 3.21], the Shapovalov scalar product $(\mathbf{n}'_\alpha, \mathbf{u}_\alpha) \in \mathbb{C}(t_1, \dots, t_N, u, v)$ equals the class $\mathfrak{J}_\alpha := [R\Gamma(\mathcal{P}^\alpha, \mathcal{O}_{\mathcal{P}^\alpha})] \in \mathrm{Frac}(K^{\hat{T}}(pt))$ up to a monomial in t, u, v . Comparing with [14, Theorem 3.1] we see that the collection of rational functions \mathfrak{J}_α is uniquely characterized by the condition $\mathfrak{J}_0 = 1$, and the recursion

relation

$$\mathfrak{J}_\alpha = \sum_{0 \leq \beta \leq \alpha} \frac{q^{(\beta, \beta)/2} z^{\beta^*}}{(q)_{\alpha - \beta}} \mathfrak{J}_\beta \quad (7.1)$$

where $q = v^2$, and $(q)_\gamma := \prod_{i=0}^{N-1} \prod_{s=1}^{c_i} (1 - q^s)$ for $\gamma = \sum_{i=0}^{N-1} c_i \alpha_i$; while $z^{\gamma^*} := \prod_{i=0}^{N-1} z_i^{c_i}$, and $z_i = t_{i+1} t_i^{-1} u^{\delta_{0,i}}$ corresponds to the highest weight of the standard Cartan generator $K_i = L_i^2 L_{i+1}^{-1} L_{i-1}^{-1} C^{\delta_{i,0}} \in \mathcal{U}_0$ (i is understood as a residue mod N in the latter formula). Now Corollary 3.7 yields the following

Corollary 7.2. *The classes $\mathfrak{J}_\alpha = [\Gamma(Z^\alpha, \mathcal{O}_{Z^\alpha})] \in \text{Frac}(K^{\widehat{T}}(pt))$ satisfy the recursion relation (7.1).*

7.3. Functions on the boundary. In case $\mathfrak{g} = \mathfrak{sl}(N)_{\text{aff}}$ we just omit the subscript \mathfrak{g} , as usually.

Proposition 7.4. $\iota_n^* : \Gamma(\partial_n Z^\alpha, \mathcal{O}_{\partial_n Z^\alpha}(-\partial_{n+1} Z^\alpha)) \hookrightarrow \Gamma(\tilde{\partial}_n Z^\alpha, \mathcal{O}_{\tilde{\partial}_n Z^\alpha}(-\partial \tilde{\partial}_n Z^\alpha))$ is an isomorphism. Equivalently, $\iota_{|\gamma|}^* : \Gamma(\partial_\gamma Z^\alpha, \mathcal{O}_{\partial_\gamma Z^\alpha}(-\partial \partial_\gamma Z^\alpha)) \hookrightarrow \Gamma(\tilde{\partial}_\gamma Z^\alpha, \mathcal{O}_{\tilde{\partial}_\gamma Z^\alpha}(-\partial \tilde{\partial}_\gamma Z^\alpha))$ is an isomorphism for any $\gamma \leq \alpha$.

Proof. Assume first that \mathfrak{g} is a simply laced finite or affine Lie algebra such that $Z_\mathfrak{g}^\alpha$ is normal for any α . We will specialize to the case $\mathfrak{g} = \mathfrak{sl}(N)_{\text{aff}}$ later on in the proof. We have $[\Gamma(\tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha, \mathcal{O}_{\tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha})] = \mathfrak{J}_{\alpha - \gamma} \cdot \frac{1}{(q)_\gamma}$ where \mathfrak{J}_β stands for the class of $[\Gamma(Z_\mathfrak{g}^\beta, \mathcal{O}_{Z_\mathfrak{g}^\beta})]$ in $\text{Frac}(K^{T \times \mathbb{C}^*}(pt))$. Also, $[\Gamma(\partial Z_\mathfrak{g}^\beta, \mathcal{O}_{\partial Z_\mathfrak{g}^\beta})] = (1 - q^{(\beta, \beta)/2} z^{\beta^*}) \mathfrak{J}_\beta$ since the (reduced) subscheme $\partial Z_\mathfrak{g}^\beta \subset Z_\mathfrak{g}^\beta$ is cut out by the equation F_β whose $T \times \mathbb{G}_m$ -degree is given by Proposition 4.4. In effect, the zero-subscheme of F_β is generically reduced (at each irreducible component) by Lemma 4.2, and hence reduced due to normality of $Z_\mathfrak{g}^\beta$. In effect, we must check that any function $f \in \Gamma(Z_\mathfrak{g}^\beta, \mathcal{O}_{Z_\mathfrak{g}^\beta})$ vanishing at the boundary is divisible by F_β . The rational function f/F_β is regular at the generic points of all the components of the boundary due to Lemma 4.2, so it is regular due to normality of $Z_\mathfrak{g}^\beta$.

All in all we see that $[\Gamma(\tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha, \mathcal{O}_{\tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha}(-\partial \tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha))]$ is equal to

$$[\Gamma(\tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha, \mathcal{O}_{\tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha})] - [\Gamma(\partial \tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha, \mathcal{O}_{\partial \tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha})] = \frac{q^{(\alpha - \gamma, \alpha - \gamma)/2} z^{(\alpha - \gamma)^*}}{(q)_\gamma} \mathfrak{J}_{\alpha - \gamma}.$$

Furthermore, since $[\Gamma(\partial_n Z_\mathfrak{g}^\alpha, \mathcal{O}_{\partial_n Z_\mathfrak{g}^\alpha}(-\partial_{n+1} Z_\mathfrak{g}^\alpha))] = [\Gamma(\partial_n Z_\mathfrak{g}^\alpha, \mathcal{O}_{\partial_n Z_\mathfrak{g}^\alpha})] - [\Gamma(\partial_{n+1} Z_\mathfrak{g}^\alpha, \mathcal{O}_{\partial_{n+1} Z_\mathfrak{g}^\alpha})]$ we have $[\Gamma(Z_\mathfrak{g}^\alpha, \mathcal{O}_{Z_\mathfrak{g}^\alpha})] = \sum_{n \geq 0} [\Gamma(\partial_n Z_\mathfrak{g}^\alpha, \mathcal{O}_{\partial_n Z_\mathfrak{g}^\alpha}(-\partial_{n+1} Z_\mathfrak{g}^\alpha))] = \sum_{\gamma \leq \alpha} [\Gamma(\partial_\gamma Z_\mathfrak{g}^\alpha, \mathcal{O}_{\partial_\gamma Z_\mathfrak{g}^\alpha}(-\partial \partial_\gamma Z_\mathfrak{g}^\alpha))]$. Let us view this equality as an equality of formal power series in q, z with *nonnegative* powers and with *nonnegative* integral coefficients. Note that $[\Gamma(\tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha, \mathcal{O}_{\tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha}(-\partial \tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha))] \succeq [\Gamma(\partial_\gamma Z_\mathfrak{g}^\alpha, \mathcal{O}_{\partial_\gamma Z_\mathfrak{g}^\alpha}(-\partial \partial_\gamma Z_\mathfrak{g}^\alpha))]$ (meaning that the LHS series is termwise bigger than or equal to the RHS series) since $\Gamma(\partial_\gamma Z_\mathfrak{g}^\alpha, \mathcal{O}_{\partial_\gamma Z_\mathfrak{g}^\alpha}(-\partial \partial_\gamma Z_\mathfrak{g}^\alpha)) \hookrightarrow \Gamma(\tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha, \mathcal{O}_{\tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha}(-\partial \tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha))$.

Finally, let us assume $\mathfrak{g} = \mathfrak{sl}(N)_{\text{aff}}$. Comparing to the equality (7.1), in view of the equality $[\Gamma(\tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha, \mathcal{O}_{\tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha}(-\partial \tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha))] = \frac{q^{(\alpha - \gamma, \alpha - \gamma)/2} z^{(\alpha - \gamma)^*}}{(q)_\gamma} \mathfrak{J}_{\alpha - \gamma}$, we must have

$[\Gamma(\tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha, \mathcal{O}_{\tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha}(-\partial \tilde{\partial}_\gamma Z_\mathfrak{g}^\alpha))] = [\Gamma(\partial_\gamma Z_\mathfrak{g}^\alpha, \mathcal{O}_{\partial_\gamma Z_\mathfrak{g}^\alpha}(-\partial \partial_\gamma Z_\mathfrak{g}^\alpha))]$ which completes the proof of the proposition. \square

Actually, the proof of Proposition 7.4 shows the converse statement as well:

Lemma 7.5. *Let \mathfrak{g} be a simply laced finite or affine Lie algebra. If $\iota_n^* : \Gamma(\partial_n Z_\mathfrak{g}^\alpha, \mathcal{O}_{\partial_n Z_\mathfrak{g}^\alpha}(-\partial \partial_n Z_\mathfrak{g}^\alpha)) \hookrightarrow \Gamma(\tilde{\partial}_n Z_\mathfrak{g}^\alpha, \mathcal{O}_{\tilde{\partial}_n Z_\mathfrak{g}^\alpha}(-\partial \tilde{\partial}_n Z_\mathfrak{g}^\alpha))$ is an isomorphism for any n , then the fermionic recursion (1.2) holds for any α .* \square

The combination of Lemma 7.5 and Lemma 6.12 implies

Theorem 7.6. *Let \mathfrak{g} be a simple simply laced Lie algebra. Then the fermionic recursion (1.2) holds for any α .* \square

8. NON SIMPLY LACED CASE

8.1. Fermionic recursion. We recall the results of [14]. Let $\check{\mathfrak{g}}$ be a simple Lie algebra with the corresponding adjoint Lie group \check{G} . Let \check{T} be a Cartan torus of \check{G} . We choose a Borel subgroup $\check{B} \supset \check{T}$. It defines the set of simple roots $\{\alpha_i, i \in I\}$. Let $G \supset T$ be the Langlands dual groups. We define an isomorphism $\alpha \mapsto \alpha^*$ from the root lattice of (\check{G}, \check{T}) to the root lattice of (G, T) in the basis of simple roots as follows: $\alpha_i^* := \check{\alpha}_i$ (the corresponding simple coroot). For two elements α, β of the root lattice of (\check{G}, \check{T}) we say $\beta \leq \alpha$ if $\alpha - \beta$ is a nonnegative linear combination of $\{\alpha_i, i \in I\}$. For such α we denote by z^{α^*} the corresponding character of T . As usually, q stands for the identity character of \mathbb{G}_m . We set $d_i = \frac{(\alpha_i, \alpha_i)}{2}$, and $q_i = q^{d_i}$. For $\gamma = \sum_{i \in I} c_i \alpha_i$, we set $(q)_\gamma := \prod_{i \in I} \prod_{s=1}^{c_i} (1 - q_i^s)$. According to [14, Theorem 3.1], the recurrence relations

$$\mathfrak{J}_\alpha = \sum_{0 \leq \beta \leq \alpha} \frac{q^{(\beta, \beta)/2} z^{\beta^*}}{(q)_{\alpha - \beta}} \mathfrak{J}_\beta$$

uniquely define a collection of rational functions \mathfrak{J}_α , $\alpha \geq 0$, on $T \times \mathbb{G}_m$, provided $\mathfrak{J}_0 = 1$. Moreover, these functions are nothing but the Shapovalov scalar products of the weight components of the Whittaker vectors in the universal Verma module over the corresponding quantum group.

8.2. Geometric interpretation. In case $\check{\mathfrak{g}}$ is simply laced, \mathfrak{J}_α is the character of $\mathbb{C}[Z_\mathfrak{g}^\alpha]$, according to Theorem 7.6. In case $\check{\mathfrak{g}}$ is not simply laced, \mathfrak{J}_α is *not* the character of $\mathbb{C}[Z_\mathfrak{g}^\alpha]$ already in the case α is a long simple coroot. We will introduce a scheme $\widehat{Z}_\mathfrak{g}^\alpha$ equipped with the action of $T \times \mathbb{G}_m$ such that the character of $\mathbb{C}[\widehat{Z}_\mathfrak{g}^\alpha]$ equals \mathfrak{J}_α .

To this end we realize $\check{\mathfrak{g}}$ as a *folding* of a simple simply laced Lie algebra $\check{\mathfrak{g}}'$, i.e. as invariants of an outer automorphism σ of $\check{\mathfrak{g}}'$ preserving a Cartan subalgebra $\check{\mathfrak{t}}' \subset \check{\mathfrak{g}}'$ and acting on the root system of $(\check{\mathfrak{g}}', \check{\mathfrak{t}}')$. In particular, σ gives rise to the same named automorphism of the Langlands dual Lie algebras $\mathfrak{g}' \supset \mathfrak{t}'$. The invariants of σ on the coroot lattice of $(\mathfrak{g}', \mathfrak{t}')$ coincide with the root lattice of $\check{\mathfrak{g}}$. Given $\alpha \geq 0$ in the root lattice of $\check{\mathfrak{g}}$, we define an automorphism ς of the based quasimaps' space $Z_\mathfrak{g}^\alpha$ as follows. It is the composition of two automorphisms: a) σ on the target; b) multiplication by ζ on the source $\mathbf{C} \cong \mathbb{P}^1$. Here ζ is

a primitive root of unity of the order equal to the order of σ . One can check that the fixed point set $(\hat{Z}_{g'}^\alpha)^\varsigma$ is connected. We define \widehat{Z}_g^α as the closure of $(\hat{Z}_{g'}^\alpha)^\varsigma$ in $Z_{g'}^\alpha$.

The equality $\mathfrak{J}_\alpha = [\mathbb{C}[\widehat{Z}_g^\alpha]]$ is proved along the lines of the argument of the previous sections. In particular, the role of the affine Grassmannian of G in the simply laced case is played by the ramified Grassmannian of (G', σ) , see [27].

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